

## ECON 507

### Problem Set 3

1. Derive the disturbance covariance matrix for the model

$$\begin{aligned}y_t &= \beta' x_t + \epsilon_t \\ \epsilon_t &= \rho \epsilon_{t-1} + u_t - \lambda u_{t-1},\end{aligned}$$

What parameter is estimated by the regression of the OLS residuals on their lagged values?

2. Consider the model

$$\begin{aligned}y_t &= \beta y_{t-1} + \epsilon_t \\ \epsilon_t &= \rho \epsilon_{t-1} + u_t.\end{aligned}$$

Show that

$$\text{plim } \frac{r}{b} = \beta \rho, \quad \text{and} \quad \text{plim } b + r = \beta + \rho,$$

where  $b$  is the OLS estimator of  $\beta$  and  $r = \sum_{t=2}^T e_t e_{t-1} / \sum_{t=1}^T e_t^2$ . If you were given only the ordinary least square regression results and the Durbin-Watson statistic, could you estimate  $\beta$  and  $\rho$ ? If so, how? If not, why not?

3. Consider the following autoregressive AR(1) process

$$u_t = \phi u_{t-1} + \epsilon_t,$$

where  $\{\epsilon_t\}$  is a white noise process and  $|\phi| < 1$ . Prove that

$$\mathbf{E}(u_t) = 0, \quad \text{and} \quad \mathbf{V}(u_t) = \sigma_u^2 = \frac{\sigma_\epsilon^2}{1 - \phi^2}, \quad \forall t.$$

4. Explain how to estimate the parameters of the following model:

$$\begin{aligned}y_t &= \beta_0 + \beta_1 x_t + \beta_2 y_{t-1} + \beta_3 y_{t-2} + e_t, \\ e_t &= \rho e_{t-1} + u_t.\end{aligned}$$

Justify why your proposed estimators are good.

5. Consider the model  $y = X\beta + u$ .

- (a) What is the formula for  $\hat{\beta}$  that minimizes  $Q(\beta) = u'Wu$ , where  $W$  is of full rank.
- (b) Show that this simplifies to the OLS estimator if  $W = I$ .
- (c) Show that this simplifies to the GLS estimator if  $W = \Omega$ .
- (d) Show that this gives the 2SLS estimator if  $W = Z(Z'Z)^{-1}Z'$ .

6. Consider the following three equation model

$$\begin{aligned} y &= \beta x + u, \\ x &= \lambda u + \epsilon, \\ z &= \gamma \epsilon + v, \end{aligned}$$

where the mutually independent errors  $u$ ,  $\epsilon$  and  $v$  are iid normal with mean 0 and variances,  $\sigma_u^2$ ,  $\sigma_\epsilon^2$  and  $\sigma_v^2$ , respectively. Show the followings:

- (a)  $\text{plim}(\hat{\beta}_{OLS} - \beta) = \lambda\sigma_u^2/(\lambda^2\sigma_u^2 + \sigma_\epsilon^2)$
- (b)  $\rho_{XZ}^2 = \gamma\sigma_\epsilon^2/(\lambda^2\sigma_u^2 + \sigma_\epsilon^2)(\gamma^2\sigma_\epsilon^2 + \sigma_v^2)$
- (c)  $\hat{\beta}_{IV} = m_{zy}/m_{zx} = \beta + m_{zu}/(\lambda m_{zu} + m_{z\epsilon})$ , where, for example,  $m_{zy} = \sum_i z_i y_i$
- (d)  $\hat{\beta}_{IV} - \beta \rightarrow 1/\lambda$  as  $\gamma \rightarrow 0$
- (e)  $\hat{\beta}_{IV} - \beta \rightarrow \infty$  as  $m_{zu} \rightarrow -\gamma\sigma_\epsilon^2/\lambda$
- (f) What do the results (d) and (e) imply regarding finite sample biases and the moments of  $\hat{\beta}_{IV} - \beta$  when the instruments are poor?