Homework 1: ECON 507

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1 Question 1

From William Greene, the restricted coefficient vector can be written as:

$$
\begin{aligned}
b* &= [I - CR]b + w && (1)\\
C &= (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1} && (2)\\
w &= Cq && (3)
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
Var(b*) &= (I - CR)\sigma^2(X'X)^{-1}(I - CR)' && (4)\\
&= \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}R(X'X)^{-1} && (5)
\end{aligned}
$$

We know $Var(b*)$ is p.s.d. We now show that $Var(b) - Var(b*) = M$ is p.s.d. Consider a quadratic form in $Var(b*)$:

$$
\begin{aligned}
z'Var(b*)z &= z'Var(b)z - \sigma^2 z(X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}R(X'X)^{-1}z && (6)\\
&= z'Var(b)z - w'[R(X'X)^{-1}R']^{-1}w && (7)
\end{aligned}
$$

where $w = \sigma[R(X'X)^{-1}z]$. Now $[R(X'X)^{-1}R']^{-1}$ is p.s.d. since its inverse is p.s.d. Thus, every quadratic form in $Var(b*)$ is less than the quadratic form in $Var(b)$ in the same vector.

2 Question 2

2.1 Derive OLS estimate for $\beta^{(2)}$ without using FWL regression

We need to do this using the partitioned inverse formula. For our regression model given, we have:

$$
\tilde{\beta} = \begin{pmatrix} \tilde{\beta}^{(1)} \\ \tilde{\beta}^{(2)} \end{pmatrix} = \begin{bmatrix} Z^{(1)'}Z^{(1)} & Z^{(1)'}Z^{(2)} \\ Z^{(2)'}Z^{(1)} & Z^{(2)'}Z^{(2)} \end{bmatrix}^{-1} \begin{bmatrix} Z^{(1)'}y \\ Z^{(2)'}y \end{bmatrix} \tag{8}
$$

Then we have that:

$$
\begin{aligned}
\tilde{\beta}^{(2)} &= -\left[Z^{(2)'}Z^{(2)} - Z^{(2)'}Z^{(1)}(Z^{(1)'}Z^{(1)})^{-1}Z^{(1)'}Z^{(2)}\right]^{-1} Z^{(2)'}Z^{(1)}(Z^{(1)'}Z^{(1)})^{-1}Z^{(1)'}y \\
&+ \left[Z^{(2)'}Z^{(2)} - Z^{(2)'}Z^{(1)}(Z^{(1)'}Z^{(1)})^{-1}Z^{(1)'}Z^{(2)}\right]^{-1} Z^{(2)'}y \\
&= \left[Z^{(2)'}Z^{(2)} - Z^{(2)'}Z^{(1)}(Z^{(1)'}Z^{(1)})^{-1}Z^{(1)'}Z^{(2)}\right]^{-1} \left[Z^{(2)'}y - Z^{(2)'}Z^{(1)}(Z^{(1)'}Z^{(1)})^{-1}Z^{(1)'}y\right] \\
&= \left(Z^{(2)'}M_1Z^{(2)}\right)^{-1} Z^{(2)'}M_1y
\end{aligned} \tag{9}
$$

2.2 Derive OLS estimator for $\beta^{(2)}$ using FWL theorem

We can now run an FWL-type regression, to again obtain an estimate for our coefficient. To do this:

1. Regress y on $Z^{(1)}$, which gives you M_1y as residuals.
2. Regress $Z^{(2)}$ on $Z^{(1)}$, which gives $M_1Z^{(2)}$ as residuals.
3. Finally, regress $M_1Z^{(2)}$ on M_1y, to obtain the coefficient $\hat{\beta}^{(2)} = \left(Z^{(2')}M_{1'}M_1Z^{(2)}\right)^{-1} Z^{(2')}M_{1'}M_1y$. However, owing to the idempotency of M_1 this is nothing but $\tilde{\beta}^{(2)}$

3 Question 3

Here, $F = \dfrac{(\hat{\beta}'X'X\hat{\beta})/k}{e'e/(n-k)}$, where $\hat{\beta} = (X'X)^{-1}X'\epsilon$, since $\beta = 0$. Now, F can be re-written as:

$$
F = \frac{\epsilon'X(X'X)^{-1}X'\epsilon/k}{\epsilon'M\epsilon/(n-k)} \tag{10}
$$

$$
= \frac{\epsilon'X(X'X)^{-1}X'\epsilon/k}{\hat{\sigma}^2} \quad \xrightarrow{p} \quad \frac{\epsilon'\epsilon}{\sigma^2} \tag{11}
$$

Further, $E(F) = \dfrac{n-k}{n-k-2}$.

4 Question 4

4.1 MLE and distribution of $\hat{\beta}$

By simply maximizing the log likelihood of the given Poisson mass function, we get $\hat{\beta}_{MLE} = \dfrac{\sum_i y_i}{\sum_i x_i}$. Now since $Y \sim$ Poisson, with $\lambda = \beta x_i$, $Z = \sum_i Y_i \sim$ Poisson with mean as $n\lambda$. Thus:

$$
f(\hat{\beta}_{MLE}) = \frac{e^{-n\beta x_i}(\beta \sum_i x_i)^{\hat{\beta}\sum_i X_i}}{\hat{\beta}\sum_i X_i!} \tag{12}
$$

Also asymptotic distribution is given by $\sqrt{n}(\hat{\beta}_{MLE} - \beta) \rightsquigarrow N(0, \frac{\beta}{X})$.

4.2 Linearity

Since Y_i has a Poisson distribution, $E(y_i) = \beta x_i$. This can be written as the model:

$$y_i = \beta x_i + \epsilon \tag{13}$$

where $E(\epsilon) = 0$. This is a linear model, as we can see, and $Var(y_i|x_i) = \lambda = \beta x_i$. Then the OLS estimator for this least squares regression is:

$$\hat{\beta}_{OLS} = \frac{\sum_i x_i y_i}{\sum_i x_i^2} \tag{14}$$

$$= \beta + \frac{\sum_i x_i \epsilon_i}{\sum_i x_i^2} \tag{15}$$

Further, using the usual assumptions, $E(X'\epsilon) = 0$, so that by the LLN, $\sum x_i \epsilon_i \xrightarrow{p} 0$, and $\hat{\beta}_{OLS} \xrightarrow{p} \beta$. Finally, $Var(\hat{\beta}_{OLS}) = (X'X)^{-1}X'\Sigma X(X'X)^{-1}$, so that in the limit, the variance becomes a "sandwich" formula, which we call $J^{-1}KJ^{-1}$. Thus, $\sqrt{n}(\hat{\beta}_{MLE} - \beta) \rightsquigarrow N(0, J^{-1}KJ^{-1})$.

4.3 Comparing asymptotic variances

$$V(\hat{\beta}_{OLS}) = (X'X)^{-1}X'\Sigma X(X'X)^{-1} = \frac{\beta \sum_i x_i^3}{(\sum_i x_i^2)^2} \tag{16}$$

Further, $AV(\hat{\beta}_{MLE}) = \dfrac{\beta}{\sum_i x_i}$. Thus, $AV(\hat{\beta}_{OLS}) - AV(\hat{\beta}_{MLE})$ is p.s.d.

5 Question 5

5.1 Find OLS estimator for β

Using the least squares procedure, we minimize the sum of squared errors with respect to β, and since we don't have an intercept here, we have that $\hat{\beta} = \dfrac{\sum_i x_i y_i}{\sum_i x_i^2}$.

5.2 Derive t-test

Consider the hypothesis $H_0 : \beta = 1$, against $H_1 : \beta \neq 1$. Then, given the above estimate of β, we have that $V(\hat{\beta}) = \dfrac{\hat{\sigma}^2}{\sum_i x_i^2}$, where $\hat{\sigma}^2 = \dfrac{\sum_i (y_i - \hat{\beta} x_i)^2}{n-1}$. Thus, we have that:

$$t = \frac{\hat{\beta} - 1}{s.e._{\hat{\beta}}} \tag{17}$$

$$= \frac{\hat{\beta} - 1}{\sqrt{\hat{\sigma}^2 / \sum_i x_i^2}} \sim t_{n-1} \tag{18}$$

5.3 Consistency

As $n \to \infty$, we have that $t \to \infty$, and probability of rejecting the null goes to 1. Thus, Power $= \Pr(Reject H_0|H_A) \to 1$, as $n \to \infty$, making the test consistent.

6 Question 6

Here, $\theta = (\beta, \sigma^2, \gamma)$. Then:

$$l(\theta) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2}\sum_i \log(\gamma' z_i)^2 - \frac{1}{2\sigma^2}\sum_i \frac{(y_i - \beta' x_i)^2}{(\gamma' z_i)^2} \tag{19}$$

We have the second derivatives, to compute the IM.

1. $\dfrac{\partial^2 l}{\partial\beta\partial\beta'} = \dfrac{1}{\sigma^2}\sum_i \dfrac{x_i x_i}{(\gamma' z_i)^2}$

2. $\dfrac{\partial^2 l}{\partial\beta\partial\sigma^2} = -\dfrac{1}{\sigma^4}\sum_i \dfrac{\epsilon_i x_i}{(\gamma' z_i)^2}$

3. $\dfrac{\partial^2 l}{\partial\beta\partial\gamma'} = -\dfrac{2}{\sigma^2}\sum_i \dfrac{\epsilon_i x_i z_i'}{(\gamma' z_i)^3}$

4. $\dfrac{\partial^2 l}{\partial(\sigma^2)^2} = \dfrac{n}{2\sigma^4} - \dfrac{1}{\sigma^6}\sum_i \dfrac{\epsilon^2}{(\gamma' z_i)^2}$

5. $\dfrac{\partial^2 l}{\partial\sigma^2\partial\gamma'} = -\dfrac{1}{\sigma^4}\sum_i \dfrac{\epsilon_i^2 z_i}{(\gamma' z_i)^3}$

6. $\dfrac{\partial^2 l}{\partial\gamma\partial\gamma'} = \sum_i \dfrac{z_i z_i'}{\gamma' z_i)^2} - \dfrac{3}{\sigma^2}\sum_i \dfrac{(y_i - \beta' x_i)^2 z_i z_i'}{(\gamma' z_i)^4}$

Based on these, you can now estimate the 3×3 information matrix, $I(\theta)$. Make sure you take the negative of the expected value of the above, in order to do this.

7 Question 7

We show this algebraically, so that we obtain an equation which says that estimates of the given parameters can be obtained using this derived equation. Note, that our OLS estimator is $\hat{\mu} = \bar{y}$, so that we can write $e_i = y_i - \bar{y} = \epsilon_i - \bar{\epsilon}$, after some algebra. Thus we can write:

$$E(e_i^2) = E(\epsilon_i^2) + E(\bar{\epsilon}^2) - 2E(\epsilon_i\bar{\epsilon}) \tag{20}$$

$$= \sigma^2 + \sigma^2\gamma^2 x_i^2 + \frac{1}{n}\left[\frac{1}{n}\sum_i \sigma_i^2 - 2\sigma_i^2\right] \tag{21}$$

In order to obtain consistent estimators for σ^2 and γ^2 from the above, by regressing least squares residuals on x_i^2, we need the last term to go to 0 in the limit. Call this term Q_n

$$\lim_{n\to\infty} Q_n = \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n}\sum_i \sigma_i^2\right] - \lim_{n\to\infty} \frac{2}{n}\sigma_i^2 \tag{22}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sigma^2 + \lim_{n\to\infty} \frac{\sigma^2\gamma^2}{n}\frac{\sum_i x_i^2}{n}(\text{why?}) \tag{23}$$

$$= \lim_{n\to\infty} \frac{\sigma^2\gamma^2}{n}\frac{\sum_i x_i^2}{n} \tag{24}$$

$$= 0 \tag{25}$$

8 Question 8

In this model, notice in the class notes, and in Question 9, that we have heteroscedasticity of a linear form. In other words, $\sigma_t^2 = \alpha_0 + \alpha_1 z_{1t} + \ldots + \alpha_p z_{pt}$. Thus, we can carry out the Breusch-Pagan test (LM test) using an artificial linear regression of squared estimated residuals on the vector $z_{t'} = (1, z_{2t}, \ldots, z_{pt})$. However, here, we have heteroscedasticity of a non-linear (exponential/multiplicative) form. We derive the LM test just like we do for the linear form, and we arrive at the same LM statistic. However, we *implement* the LM test using a log-linear regression.

8.1 Deriving the LM test

Consider the log-likelihood of this model.

$$l(\theta) = \text{const.} - \frac{1}{2}\sum_t \log\sigma_t^2 - \frac{1}{2}\sum_t \frac{(y_t - x_t'\beta)^2}{\sigma_t^2} \tag{26}$$

$$= \text{const.} - \frac{1}{2}\sum_t z_t'\alpha - \frac{1}{2}\frac{u_t^2}{e^{z_t'\alpha}} \tag{27}$$

Then, we can write the score vector as:

$$\frac{\partial l}{\partial \alpha} = \frac{1}{2}\sum_t z_t\left[\frac{u_t^2}{\sigma_t^2} - 1\right] \tag{28}$$

Since this is an LM test, evaluated only under the null, we're only interested in the second derivative of the score with respect to α. Thus we have:

$$\frac{\partial^2 l}{\partial\alpha\alpha'} = -\frac{1}{2}\sum_t z_t z_t' \frac{u_t^2}{\sigma_t^2} \tag{29}$$

Further, we have:

$$E\left[-\frac{\partial^2 l}{\partial\alpha\alpha'}\right] = \frac{1}{2}\sum_t z_t z_t' \tag{30}$$

$$= \frac{Z'Z}{2} \tag{31}$$

where the last step follows from the fact that under $H_0 : \sigma_t^2 = \alpha_1 = \sigma^2$, and so $E(u_t^2) = \sigma^2$. Also, under the null:

$$\left.\frac{\partial l}{\partial \alpha}\right|_{\tilde{\theta}} = \frac{1}{2}\sum_t z_t \left[\frac{\hat{u}_t^2}{\hat{\sigma}^2} - 1\right] \tag{32}$$

where we will call $\frac{\hat{u}_t^2}{\hat{\sigma}^2} = \hat{r}_t$. Then, the LM statistic can be written as:

$$LM_H = \frac{1}{2}(\hat{r} - \mathbf{1})' Z(Z'Z)^{-1} Z'(\hat{r} - \mathbf{1}) \tag{33}$$

which is the same as the LM test derived for the linear form of heteroscedasticity. You can further derive the artificial regression form of it by using the same procedure, as for the linear form (See pg. 74-75 in class notes). However, implementation of the test is done using regression of $\log(\hat{u}_t^2$ on $z_{t'} = (1, z_{2t}, \ldots, z_{pt})$, which is basically the matrix Z.

9 Question 9

This is from your class notes, Pg. 72-75, Breusch-Pagan test, for linear form of heteroscedasticity.