University of Illinois
Spring 2010

Dept. of Economics
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ECON 507

Problem Set 2

1. Prove the result that the restricted least squares estimator never has a larger covariance matrix that the unrestricted least square estimator.

2. Consider the regression model

$$\begin{aligned} y &= Z\beta + u \\ &= Z^{(1)}\beta^{(1)} + Z^{(2)}\beta^{(2)} + u \end{aligned}$$

where $Z = [Z^{(1)}, Z^{(2)}]$ and $\beta = (\beta^{(1)}, \beta^{(2)})'$ are partitioned in a matching fashion.

   (a) Derive $\hat{\beta^{(2)}}_n$ with $M_1 = I - Z^{(1)}(Z^{(1)}Z^{(1)})^{-1}Z^{(1)}$.

   (b) (i) Regress y on $Z^{(1)}$ and obtain the residual u_1; (ii) Regress each column of $Z^{(2)}$ on $Z^{(1)}$ to obtain the residual matrix $\mathbf{U}_2$ whose column are the residuals of each regression; (iii) Regress u_1 on $\mathbf{U}_2$, and check whether the OLS estimator is the same as (a).

3. For the classical normal regression model $y = X\beta + \epsilon$ with no constant term and k regressors, what is

$$\text{plim}F[K, n-k] = \text{plim}\frac{R^2/K}{(1-R^2)/(n-K)}$$

assuming that the true value of β is zero? What is the exact expected value?

4. Consider y_i is a discrete random variable whose distribution given x_i is

$$\text{Prob}(Y_i = y_i) = \frac{e^{-\beta x_i}(\beta x_i)^{y_i}}{y_i!}, \quad y_i = 0, 1, \cdots$$

Assume that $x_i > 0$ for all i.

   (a) Derive the maximum likelihood estimator of β and its asymptotic distribution. Note that y_i has a Poisson distribution. Obtain the exact distribution of the maximum likelihood estimator.

(b) Check whether this model is linear model. What is the conditional variance of y_i given x_i? Show that the least square estimator is consistent. Derive the asymptotic distribution of the estimator. What assumption are necessary to show asymptotic normality?

(c) Prove that the asymptotic variance of the maximum likelihood estimator is small than that of the least square estimator.

5. We have the simple regression model, $y_i = \beta x_i + \epsilon_i$, $\epsilon_i \sim N(0, \sigma^2)$, $i = 1, 2, \cdots, n/$

   (a) Find OLS estimator of β.

   (b) Derive the t-test for H_0:$\beta = 1$ against H_1:$\beta \neq 1$.

   (c) Show that the above test is consistent.

6. Derive the log-likelihood function, first-order conditions for maximization, and information matrix for the model

$$\begin{aligned} y_i &= \beta' x_i + \epsilon_i, \\ \epsilon_i &\sim N(0, \sigma^2(\gamma' z_i)^2). \end{aligned}$$

7. Suppose that the regression model is

$$y_i = \mu + \epsilon_i,$$

   where $\epsilon_i \sim N(0, \sigma^2(1 + (\gamma x)^2))$. Show that σ^2 and γ^2 can be consistently estimated by a regression of the least squares residuals on a constant and x^2. Can you say something further about this estimator?

8. Consider the model

$$y_t = x_t' \beta + u_t,$$

   where $x_t' = (1, x_{2t}, x_{3t}, \cdots, x_{kt})$, and u_t follows normal distribution. Instead of the usual assumption of homoscedasticity, we assume

$$\begin{aligned} Eu_t &= 0 \qquad \text{for all } t \\ \sigma_t^2 &= Eu_t^2 = e^{z_t' \alpha}, \end{aligned}$$

   where $z_t' = (1, z_{2t}, z_{3t}, \cdots, z_{pt})$ is a vector of known variables, possibly including some of the x variables or functions of the x variables, and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_p)$ is a vector of unknown parameters. The null hypothesis of homoscedasticity takes the form, $H_0 : \alpha_2 = \alpha_3 = \cdots = \alpha_p = 0$. Derive the LM test for heteroscedasticity.

9. Take the model in question 8 and assume that heteroskedasticity has the form

$$\sigma_t^2 = \alpha_0 + \alpha_1 z_{1t} + \cdots + \alpha_p z_{pt}.$$

Derive the LM test for $H_0 : \alpha_1 = \cdots = \alpha_p = 0$ and compare the test statistics.

10. [Butler, McDonald, Nelson and White (1990)] Let us consider two additional distributions for u_t:

$$f(u) = \frac{1}{2^{1+1/\delta}\Gamma(1+\frac{1}{\delta})}\frac{1}{\sigma}\exp\left(-\frac{1}{2}\left|\frac{u}{\sigma}\right|^{\delta}\right), \quad 0 < \delta \leq \infty,$$

and t-distribution with γ degree of freedom. Consider the following regression equation

$$(R_i - R_f)_t = \beta_{1i} + \beta_{2i}(R_m - R_f)_t + u_{it},$$

where R_i: the rate of return on equity for firm i; R_m: the rate of return on the market portfolio; R_f: the risk-free rate; u_{it}: a white noise random variable.

(a) Test for normality of u_i.

(b) Assume that the true density is given as the above (two densities). Estimate all the parameters using maximum likelihood method. And compare the results with Tables 1-2.

(c) Calculate sample mean, variance, skewness and kurtosis of the residuals from (b). And compare these statistics with implied moments of the estimated densities. Are they close enough?

(d) The above densities are symmetric ones. Suppose your younger brother wants to use asymmetric densities to estimate parameters. Give him good comments. If possible, estimate the parameters using the proposed asymmetric density.