Homework 1: ECON 507

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March 19, 2010

1 Question 1

After expansion we get: $X = (F')^{-1}(E')^{-1}B'A' + H'G'B'A' + (F')^{-1}(E')^{-1}CD + H'G'CD$.

2 Question 2

We know that the RHS is:

$$
\begin{aligned}
X'M_0X + n(\bar{x}-a)(\bar{x}-a)' &= X\left[I - \frac{\mathbf{11'}}{n}\right]X + n(\bar{x}-a)(\bar{x}-a)' && (1)\\
&= \sum_i (x_i-\bar{x})(x_i-\bar{x})' + n(\bar{x}-a)(\bar{x}-a)' && (2)\\
&= \sum_i x_i^2 - 2n\bar{x}a + na^2 && (3)
\end{aligned}
$$

However since $\sum_i x_i = n\bar{x}$, the LHS is also the same as above.

3 Question 3

When you write out the Kronecker products and their traces, you get:

$$
\begin{aligned}
tr(A \otimes B) = tra_{ij}b_{ij} &= \sum_i^n a_{ii}b_{ii} && (4)\\
tr(A)tr(B) = \sum_i^n a_{ii}\sum_i^n b_{ii} &= \sum_i^n a_{ii}b_{ii} && (5)
\end{aligned}
$$

4 Question 4

The eigenvalue equation you get is: $-\lambda(\lambda^2 - 15\lambda + 5) = 0$. The eigenvalues you obtain are: $\lambda = 0, \frac{15\pm\sqrt{205}}{2}$.

5 Question 5

Using the quotient and chain rule:

$$\frac{\partial(\mathbf{x}'\mathbf{A}\mathbf{x}/\mathbf{x}'\mathbf{B}\mathbf{x})}{\partial z} = \frac{\frac{\partial \mathbf{x}}{\partial z}\mathbf{x}(\mathbf{A}+\mathbf{A}')(\mathbf{x}'\mathbf{B}\mathbf{x}) + \frac{\partial \mathbf{x}}{\partial z}\mathbf{x}(\mathbf{B}+\mathbf{B}')(\mathbf{x}'\mathbf{A}\mathbf{x})}{(\mathbf{x}'\mathbf{B}\mathbf{x})^2} \tag{6}$$

6 Question 6

6.1 Getting the matrices

$$P = \frac{1}{108}\begin{bmatrix} 59 & 11 & 51 & -13 \\ 11 & 35 & 15 & 47 \\ 51 & 15 & 45 & -3 \\ -13 & 47 & -3 & 77 \end{bmatrix} \tag{7}$$

$$M = \frac{1}{108}\begin{bmatrix} 49 & -11 & -51 & 13 \\ -11 & 73 & -15 & -47 \\ -51 & -15 & 63 & 3 \\ 13 & -47 & 3 & 31 \end{bmatrix} \tag{8}$$

MP is a 4×4 matrix of zeroes. Finally, P and M based on XQ are the same as above.

6.2 Eigenvalues and Eigenvectors

Both M and P being idempotent, have only eigenvalues 0 and 1.

7 Question 7

For eigenvalues of A, the equation is:

$$(1 - \lambda + 3\rho)(1 - \lambda - \rho)^3 = 0 \tag{9}$$

which gives eigenvalues $\lambda_1 = 1 + 3\rho$ and $\lambda_j = 1 - \rho$, $j = 2, 3, 4$.
For λ_1, eigenvector is $(1, 1, 1, 1)'$ and for the other eigenvalues, the eigenvector is $(1, 1, -1, -1)'$.

8 Question 8

Since both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased, we have that $E(\hat{\theta}) = \theta$. This implies that $c_1 + c_2 = 1$. Call $c_1 = c$, and $c_2 = 1 - c$. Call $Var(\hat{\theta}) = \upsilon$, $Var(\hat{\theta}_1) = \upsilon_1$ and $Var(\hat{\theta}_2) = \upsilon_2$. Then, the variance can be written as:

$$\upsilon = c^2\upsilon_1 + (1 - c^2)\upsilon_2 \tag{10}$$

We can now minimize this variance w.r.t. c, and solve for c. We thus obtain $c_1 = \dfrac{\upsilon_2}{\upsilon_1 + \upsilon_2}$, and $c_2 = \dfrac{\upsilon_1}{\upsilon_1 + \upsilon_2}$. On taking the second order derivative, we see that indeed this is the minimu. Thus, the minimized variance is $\upsilon^* = \dfrac{\upsilon_1\upsilon_2}{\upsilon_1 + \upsilon_2}$.

9 Question 9

Suppose the constant term is written as $a = \sum_i d_i y_i = \alpha \sum_i d_i + \beta \sum_i d_i x_i$. Then for a to be unbiased, we need $\sum_i d_i = 1$, and $\sum_i d_i x_i = 0$. On minimizing $Var(a)$, subject to these two above constraints, we get as solutions:

$$\lambda_1 = -2\sigma^2 \frac{\sum_i x_i^2}{nS_{xx}} \tag{11}$$

$$\lambda_2 = 2\sigma \frac{2\bar{x}}{S_{xx}} \tag{12}$$

$$d_i = \frac{\sum_i x_i^2 - (\sum_i x_i)x_i}{nS_{xx}} \tag{13}$$

Then given the above, $a = \sum_i d_i y_i = \bar{y} - \frac{\sum_i x_i y_i - n\bar{x}\bar{y}}{S_{xx}} = \bar{y} - \hat{\beta}\bar{x}$, making the estimator unbiased.

10 Question 10

The least squares estimator for a model without the constant is $\tilde{\beta} = \frac{\mathbf{x}'\mathbf{y}}{\mathbf{x}'\mathbf{x}}$, whereas for the model with a constant, we have $\hat{\beta} = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}$. The ratio of their variances is:

$$Var(\tilde{\beta})/Var(\hat{\beta}) = [\sigma^2/\mathbf{x}'\mathbf{x}]/[\sigma^2/\sum_i (x_i - \bar{x})^2] \leq 1 \tag{14}$$

since $\sum_i (x_i - \bar{x})^2 = \mathbf{x}'\mathbf{x} + n\bar{x}^2$.

11 Question 11

Consider the two models: unrestricted model, with a constant, and the restricted model (indexed by R) without a constant.

In the unrestricted model, we minimize the sum of squared residuals, to get coefficient vector $\hat{\beta} = (X'X)^{-1}X'y$, and residuals $\hat{\epsilon}$. In the restricted model, we minimize sum of squared residuals, with restriction $R\beta = \gamma$, where $= (1, 0, \ldots, 0)'$, and $\gamma = 0$. When we solve the restricted problem, we get, as solution:

$$\begin{pmatrix} \tilde{\beta} \\ \tilde{\lambda} \end{pmatrix} = \begin{pmatrix} X'X & R' \\ R & 0 \end{pmatrix}^{-1} \begin{pmatrix} X'y \\ 0 \end{pmatrix} \tag{15}$$

where λ is simply the Lagrange multiplier. Solving this out:

$$\tilde{\beta} = \hat{\beta} - (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}R\hat{\beta} \tag{16}$$

Also, since we can write

$$\tilde{\epsilon} = \hat{\epsilon} - X(\tilde{\beta} - \hat{\beta}) \tag{17}$$

we can derive:

$$\tilde{\epsilon}'\tilde{\epsilon} = \hat{\epsilon}'\hat{\epsilon} + \hat{\beta}'R' \underbrace{[R(X'X)^{-1}R']^{-1}}_{\geq 0} R\hat{\beta} \tag{18}$$

From the above, we thus have that $\tilde{\epsilon}'\tilde{\epsilon} \geq \hat{\epsilon}'\hat{\epsilon}$, which implies that restricted R^2 is less than the unrestricted R^2.

12 Question 12

We can write e_i as $e_i = y_i - x_i'\hat{\beta}$, so that $e_i - \epsilon_i = x_i'(\beta - \hat{\beta})$. Since plim$\hat{\beta} = \beta$, we have the result.

13 Question 13

1. Coefficient from the original untransformed regression: $\hat{\beta}^{(1)} = (X'M_0'X)^{-1}X'M_0'y$
2. Transform both y and X: $\hat{\beta}^{(2)} = (X'M_0'M_0X)^{-1}X'M_0'M_0y = \hat{\beta}^{(1)}$
3. Transform only X: $\hat{\beta}^{(3)} = (X'M_0'M_0X)^{-1}X'M_0'y = \hat{\beta}^{(1)}$
4. Transform only y: $\hat{\beta}^{(4)} = (X'X)^{-1}X'M_0'y \neq \hat{\beta}^{(1)}$

14 Question 14

Here, $F = \dfrac{(\hat{\beta}'X'X\hat{\beta})/k}{e'e/(n-k)}$, where $\hat{\beta} = (X'X)^{-1}X'\epsilon$, since $\beta = 0$. Now, F can be re-written as:

$$F = \frac{\epsilon'X(X'X)^{-1}X'\epsilon/k}{\epsilon'M\epsilon/(n-k)} \tag{19}$$

$$= \frac{\epsilon'X(X'X)^{-1}X'\epsilon/k}{\hat{\sigma}^2} \quad \xrightarrow{p} \quad \frac{\epsilon'\epsilon}{\sigma^2} \tag{20}$$

Further, $E(F) = \dfrac{n-k}{n-k-2}$.

15 Question 15

The example model you have here is:

$$y_i = \theta_1 + u_i \tag{21}$$

where $u_i \sim N(0, \theta_2)$ and are i.i.d., $i = 1, \ldots, n$. Thus, $y_i \sim N(\theta_1, \theta_2)$, and are also i.i.d. You need to use this model to construct the IM test.

15.1 Log-likelihood

$$f(y_i) = \frac{1}{\sqrt{2\pi}\sqrt{\theta_2}} e^{\frac{-(y_i-\theta_1)^2}{2\theta_2}} \tag{22}$$

Then:

$$l(\theta) = \log f(y_i) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log\theta_2 - \frac{1}{2}\frac{(y_i-\theta_1)^2}{\theta_2} \tag{23}$$

15.2 Vech component

We know:

$$\hat{a}_n(\hat{\theta}_n) = \frac{1}{n}\sum_{i=1}^{n} a(y_i, \hat{\theta}_n) \tag{24}$$

This is a "vech", or a $q \times 1$ vector created from $C(\theta)_{p\times p}$. Here, $p = 2 \Rightarrow q \leq 3$.

For creating $\hat{a}_n(\hat{\theta}_n)$, we need:

1. $\dfrac{\partial \log f}{\partial \theta_1} = \dfrac{y_i - \theta_1}{\theta_2}$
2. $\dfrac{\partial \log f}{\partial \theta_2} = \dfrac{(y_i - \theta_1)^2}{2\theta_2^2} - \dfrac{1}{2\theta_2}$
3. $\dfrac{\partial^2 \log f}{\partial \theta_1^2} = -\dfrac{1}{\theta_2}$
4. $\dfrac{\partial^2 \log f}{\partial \theta_2^2} = \dfrac{1}{2\theta_2^2} - \dfrac{(y_i - \theta_1)^2}{\theta_2^3}$

Then compute squares and cross-products $i, j = 1, 2$. Then create $a\left(y_i, \hat{\theta}_n\right)$ using given formula:

$$a\left(y_i, \hat{\theta}_n\right) = \begin{bmatrix} \left(\frac{\partial l(\theta)}{\partial \theta_1}\right)^2 + \frac{\partial^2 l(\theta)}{\partial \theta_1^2} \\ \frac{\partial l(\theta)}{\partial \theta_1}\frac{\partial l(\theta)}{\partial \theta_2} + \frac{\partial^2 l(\theta)}{\partial \theta_1 \partial \theta_2} \\ \left(\frac{\partial l(\theta)}{\partial \theta_2}\right)^2 + \frac{\partial^2 l(\theta)}{\partial \theta_2^2} \end{bmatrix} = \begin{bmatrix} \frac{(y_i-\theta_1)^2}{\theta_2^2} - \frac{1}{\theta_2} \\ \frac{(y_i-\theta_1)^3}{2\theta_2^3} - \frac{3(y_i-\theta_1)}{2\theta_2^2} \\ \frac{3}{4\theta_2^2} - \frac{3(y_i-\theta_1)^2}{2\theta_2^3} + \frac{(y_i-\theta_1)^4}{4\theta_2^4} \end{bmatrix} \tag{25}$$

$$\hat{a}_n(\hat{\theta}_n) = \frac{1}{n}\sum_{i=1}^{n} a\left(y_i, \hat{\theta}_n\right) = \begin{bmatrix} 0 \\ \frac{1}{2n\hat{\theta}_2^3}\sum_i \hat{u}_i^3 \\ \frac{1}{4n\hat{\theta}_2^4}\sum_i \left(\hat{u}_i^4 - 3\hat{\theta}_2^2\right) \end{bmatrix} \tag{26}$$

where $\hat{\theta}_2 = \dfrac{\sum_i \hat{u}_i^2}{n}$, and $\hat{u}_i = y_i - \hat{\theta}_1$. Also, notice that:

$$\hat{a}_n(\hat{\theta}_n) = \frac{1}{n}\sum_{i=1}^{n} a\left(y_i, \hat{\theta}_n\right) = \begin{bmatrix} 0 \\ \left(2n\hat{\theta}_2^{\frac{3}{2}}\right)^{-1} \times \text{Skewness} \\ \left(4n\hat{\theta}_2^2\right)^{-1} \times \text{Excess Kurtosis} \end{bmatrix} \tag{27}$$

15.3 Variance

Now consider assembling components of the Variance.

$$A(\theta_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E\left[\frac{\partial a(\theta_0)}{\partial \theta_0}\right] \tag{28}$$

$$A(\theta_0) = \begin{bmatrix} 0 & -\frac{1}{\theta_2^2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{29}$$

Similarly:

$$J(\theta_0) = \begin{bmatrix} \frac{1}{\theta_2} & 0 \\ 0 & \frac{1}{2\theta_2^2} \end{bmatrix} \tag{30}$$

Both $A(\theta_0)$ and $J(\theta_0)$ can be consistently estimated using $\hat{\theta}$. Next, define

$$g_i(\hat{\theta}) = a\left(y_i, \hat{\theta}_n\right) + A(\hat{\theta})J(\hat{\theta})^{-1}\frac{\partial l(\hat{\theta})}{\partial \theta} \tag{31}$$

$$= (\hat{g}_{i1}, \hat{g}_{i2}, \hat{g}_{i3})' \tag{32}$$

Notice that:

$$A(\hat{\theta})J(\hat{\theta})^{-1} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{33}$$

Further:

$$A(\hat{\theta})J(\hat{\theta})^{-1}\frac{\partial l(\hat{\theta})}{\partial \theta} = \begin{bmatrix} -\frac{1}{\hat{\theta}_2} + \frac{1}{\hat{\theta}_2^2}(y_i - \hat{\theta}_1)^2 \\ 0 \\ 0 \end{bmatrix} \tag{34}$$

Thus:

$$g_i(\hat{\theta}) = \begin{bmatrix} 0 \\ -\frac{3}{2\hat{\theta}_2^2}(y_i - \hat{\theta}_1) + \frac{1}{2\hat{\theta}_2^3}(y_i - \hat{\theta}_1)^3 \\ \frac{1}{4\hat{\theta}_4^2}(y_i - \hat{\theta}_1)^3 - \frac{3}{2}\frac{1}{\hat{\theta}_2^3}(y_i - \hat{\theta}_1)^2 + \frac{3}{4\hat{\theta}_2^2} \end{bmatrix} \tag{35}$$

Now consider the whole variance equation:

$$V(\theta_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E\left[g_i(\theta_0)g_i(\theta_0)'\right] \tag{36}$$

where

$$g_i(\theta_0)g_i(\theta_0)' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & g_{i2}g_{i2} & g_{i2}g_{i3} \\ 0 & g_{i3}g_{i2} & g_{i3}g_{i3} \end{bmatrix} \tag{37}$$

Now consider it component by component:

$$E[g_{i2}g_{i2}] = E\left[\frac{1}{4\theta_2^6}(y_i-\theta_1)^6 + \frac{9}{4\theta_2^4}(y_i-\theta_1)^2 - \frac{3}{2\theta_2^5}(y_i-\theta_1)^4\right] \tag{38}$$

$$= \frac{3}{2}\frac{1}{\theta_2^3} \tag{39}$$

Also:

$$E[g_{i2}g_{i3}] = E[g_{i3}g_{i2}] = 0 \tag{40}$$

$$\begin{aligned} E[g_{i3}g_{i3}] =& E\left[\frac{1}{16\theta_2^8}(y_i-\theta_1)^8 + \frac{9}{4\theta_2^6}(y_i-\theta_1)^4 + \frac{9}{16\theta_2^4} - \frac{3}{4\theta_2^7}(y_i-\theta_1)^6\right. \\ & \left. -\frac{9}{4\theta_2^5}(y_i-\theta_1)^2 + \frac{3}{8\theta_2^6}(y_i-\theta_1)^4\right] \end{aligned} \tag{41}$$

By using normality, $E(y_i-\theta_1)^2 = \theta_2$, $E(y_i-\theta_1)^4 = 3\theta_2^2$, $E(y_i-\theta_1)^6 = 15\theta_2^3$, $E(y_i-\theta_1)^8 = 105\theta_2^4$. Thus:

$$E[g_{i3}g_{i3}] = \frac{3}{2}\frac{1}{\theta_2^4} \tag{42}$$

Finally, we have:

$$V(\theta_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2}\frac{1}{\theta_2^3} & 0 \\ 0 & 0 & \frac{3}{2}\frac{1}{\theta_2^4} \end{bmatrix} \tag{43}$$

This can be consistently estimated using $V(\hat{\theta})$. Then, finally, combining all the above, we have:

$$IM = n\hat{a}_n(\hat{\theta}_n)'V(\hat{\theta})^{-1}\hat{a}_n(\hat{\theta}_n) \tag{44}$$

$$= n\left(\frac{\beta_1}{6} + \frac{(\beta_2-3)^2}{24}\right) \sim \chi_2^2 \tag{45}$$