Xiamen University | **Econometrics II** | **WISE**

Spring 2008 | | **Sung Yong Park**

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First Examination
Monday, April 14, 2008
2:30 - 5:30 pm, 105 QunXian II.

Name: ____

ID# : ____

SOLUTION

This solution contains only answers. Your exam answers will be graded based on this solution, and they will be expected, at least, to have points along with the presented solution.

This is a closed-book examination. Answer as many questions as you can. Credit for each question is given in [ ]. GOOD LUCK!

1. [15]
Note that $\hat{\beta} = \sum x_i y_i / \sum x_i^2$, $V(\hat{\beta}) = \sigma^2/\sum x_i^2$, and $\hat{\sigma}^2 = (1/(n-1))\sum(y_i - \hat{\beta}x_i)^2$. The t-statistic is

$$t = \frac{\hat{\beta}-1}{\text{s.e}\hat{\beta}} = \frac{\hat{\beta}-1}{\sqrt{\hat{\sigma}^2/\sum x_i^2}} = \frac{\sqrt{\sum x_i^2}(\hat{\beta}-1)}{\hat{\sigma}} \sim t_{n-1}. \quad (1)$$

And note that $\partial l(\theta)/\partial\beta = (\sum y_i x_i - \beta \sum x_i^2)/\sigma^2$. Since $\tilde{\beta} = 1$ and $\tilde{\sigma}^2 = (1/n)\sum(y_i - x_i)^2$, $d(\tilde{\beta}) = (n\sum y_i x_i - \sum x_i^2)/\sum(y_i - x_i)^2$, and $I(\tilde{\beta})^{-1} = \tilde{\sigma}^2/\sum x_i^2 = (1/n)\sum(y_i - x_i)^2/\sum x_i^2$. Thus,

$$\text{LM} = n\frac{[\sum y_i x_i - \sum x_i^2]^2}{\sum(y_i - x_i)^2 \sum x_i^2} \quad (2)$$

The relationship can be found from the equation (after tedious calculation)

$$\frac{t^2/(n-1)}{1+t^2/(n-1)} = \frac{[\sum y_i x_i - \sum x_i^2]^2}{\sum(y_i - x_i)^2 \sum x_i^2}. \quad (3)$$

From (2) and (3)

$$\text{LM} = n\frac{t^2/(n-1)}{1+t^2/(n-1)}.$$

2. [10+7]
a. Note that: OLSE from (1) is $\hat{\beta}_1 = (X_1'M_2X_1)^{-1}X_1'M_2Y$, and OLSE from (2) is $\check{\beta}_1 = (X_1'X_2)^{-1}X_1'Y$.

Unbiasedness:

$\mathbb{E}(\hat{\beta}_1) = (X_1'M_2X_1)^{-1}X_1'M_2\mathbb{E}(Y) = (X_1'M_2X_1)^{-1}X_1'M_2X_1\beta_1 + (X_1'M_2X_1)X_1'M_2X_2\beta_2 = \beta_1$ (since $M_2X_2 = 0$). Therefore, $\hat{\beta}_1$ is always unbiased.
Now, $\mathbb{E}(\check{\beta}_1) = (X_1'X_1)^{-1}X_1'(X_1\beta_1 + X_2\beta_2) = \beta_1 + (X_1'X_1)^{-1}X_1'X_2\beta_2$. Therefore, $\mathbb{E}(\check{\beta}_1) = \beta_1$ only if $\beta_2 = 0$ (i.e., models (1) and (2) are the same) and/or $X_1'X_2 = 0$ (i.e., X_1 and X_2 matrices are orthogonal).

Efficiency:

We can write $\hat{\beta}_1 = \beta_1 + (X_1'M_2X_1)^{-1}X_1'M_2\epsilon$.
Then, $V(\beta_1) = (X_1'M_2X_1)^{-1}X_1'M_2V(\epsilon)M_2X_1'(X_1, M_2X_1)^{-1} = \sigma^2(X_1'M_2X_1)^{-1}$. We also have $V(\check{(\beta)}_1) = \sigma^2(X_1'X_1)^{-1}$.
We have $V(\check{\beta}_1)^{-1} - V(\hat{\beta}_1)^{-1} = (1/\sigma^2)(X_1'X_2(X_2'X_2)^{-1}X_2'X_1)$ (p.s.d), and $V(\hat{\beta}_1) -$

$V(\breve{\beta}_1)$ is p.s.d. Therefore the $\breve{\beta}_1$ is *always* more efficient than the $\hat{\beta}_1$ (whether or nor $\beta_2 = 0$).

b. We can avoid the use of partitioned inverse formula. Let us denote $X = [X_1, X_2]$ and $\beta = (\beta_1, \beta_2)'$. Then (1) is simply, $Y = X\beta + \epsilon$. Then $\hat{\beta}_{OLS} = (\hat{\beta}_1, \hat{\beta}_2)' = (X'X)^{-1}X'Y$.
Now, $\mathbb{E}(\hat{\beta}_{OLS}) = (X'X)^{-1}X'\mathbb{E}(Y) = (X'X)^{-1}X'X_1\beta_1 = (X'X)^{-1}X'(X_1, X_2)(\beta_1, 0)' = (\beta_1, 0)'$. Therefore, $\mathbb{E}(\hat{\beta}_1) = \beta_1$ and $\mathbb{E}(\hat{\beta}_2) = 0$. Hence, the OLS estimator of β_1 from the model (1) is unbiased.

3. [5+5]
Variance covariance matrix of W_i is

$$V[W_i] = \begin{bmatrix} V(w_{i1}) & Cov(w_{i1}, w_{i2}) & \cdots & Cov(w_{i1}, w_{iT}) \\ & V(w_{i2}) & \cdots & Cov(w_{i2}, w_{iT}) \\ & & \ddots & \vdots \\ & & & V(w_{iT}) \end{bmatrix}$$

Now $V(w_{it}) = V(u_i + \epsilon_{it}) = V(u_i) + V(\epsilon_{it}) = \sigma_u^2 + \sigma_\epsilon^2 = 1$ (independence). And $Cov(w_{it}, w_{it'}) = V(u_i) = \sigma_u^2$ (independence). Hence, we can write

$$V(W_i) = (1 - \sigma_u^2)I + \sigma_u^2 \mathbf{1}\mathbf{1}'.$$

4. [(6+3)+(5+4)]

1. Let MLE $\tilde{\beta}$ and $\tilde{\sigma}^2$. Then, $\tilde{\beta} = (X'X)^{-1}X'Y$ and $\tilde{\sigma}^2 = (1/n)(Y - X\tilde{\beta})'(Y - X\tilde{\beta})$. We know that OLS estimator are: $\hat{\beta} = (X'X)^{-1}(X'Y)$ and $\hat{\sigma}^2 = (1/(n-k))(Y - X'\hat{\beta})'(Y - X'\hat{\beta})$. Note that the MLE $\tilde{\beta}$ is same as the OLS estimator. MLE $\tilde{\sigma}^2$ and OLS $\hat{\sigma}^2$ differ only by their denominator.

2.a. Basically,

$$l(\theta) = K - n \log \phi - \frac{1}{2} \sum \frac{|y_i - x_i'\beta|^\delta}{\phi^\delta},$$

where $\theta = (\beta', \phi)'$. The MLE of β is

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} |y_i - x_i'\beta|^\delta.$$

2.b. (i) For $\delta = 1$, we have $\hat{\beta}_{MLE}$ = median of $(y_1, y_2, \cdots, y_n)$; (ii) For $\delta = 2$, we have $\hat{\beta}_{MLE} = \sum y_i/n$; (iii) For $\delta \to \infty$, obviously any value of β can minimize $\sum(y_i - \beta)^\infty$. But, note that as $\delta \to \infty$, $f(\epsilon_i)$ becomes a uniform distribution. One possibly use $\hat{\beta} = (y_{(n)} - y_{(1)})/2$, where $y_{(j)}$ is j-th order statistic for y.

5. [5+5+5]

a. $l(\theta) = \log f(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = \log \prod_{i=1}^n f(\epsilon_i) = \sum_{i=1}^n \log f(\epsilon_i)$, where $\epsilon_i = y_i - x_i'\beta$.

$$\log f(\epsilon_i) = K_1 + \int \frac{c_1 - \epsilon_i}{\sigma^2 - c_1\epsilon_i} d\epsilon_i,$$

where K_1 is a constant of integration. Or

$$\begin{aligned} f(\epsilon_i) &= K_2 \exp\left[\int \frac{c_1 - \epsilon_i}{\sigma^2 - c_1\epsilon_i} d\epsilon_i\right], \\ &= K_2\psi(\epsilon_i) \end{aligned}$$

where $\psi(\epsilon_i) = \exp(\int (c_1 - \epsilon_i)/(\sigma^2 - c_1\epsilon_i) d\epsilon_i)$ and $K_2 = e^{K_1} = 1/(\int \psi(\epsilon_i) d\epsilon_i)$. Thus, $f(\epsilon_i) = \psi(\epsilon_i)/\int \psi(\epsilon_i) d\epsilon_i$.

$$l(\theta) = \sum \log \psi(\epsilon_i) - n \log \int \psi(\epsilon_i) d\epsilon_i,$$

where $\theta = (\beta', \sigma^2, c_1)'$.

b. For the normal distribution, $\partial \log f(\epsilon_i)/\partial \epsilon_i$ is linear in ϵ_i. This can be done by putting $c_1 = 0$. Therefore, a test for normality can be obtained by testing $H_0 : c_1 = 0$. An score test for normality is based on $\partial l(\tilde{\theta})/\partial c_1$ evaluated at $c_1 = 0$.

$$\frac{\partial l(\theta)}{\partial c_1} = \sum \frac{\partial \log \psi(\epsilon_i)}{\partial c_1} - n \frac{\int \frac{\partial \psi(\epsilon_i)}{\partial c_1} d\epsilon_i}{\int \psi(\epsilon_i) d\epsilon_i} \tag{4}$$

First term in (4):

$$\sum \frac{\partial}{\partial c_1} \int \frac{c_1 - \epsilon_i}{\sigma^2 - c_1\epsilon_i} d\epsilon_i = \sum \int \frac{(\sigma^2 - c_1\epsilon_i) - (c_1 - \epsilon_i)(-\epsilon_i)}{(\sigma^2 - c_1\epsilon_i)^2} d\epsilon_i.$$

This reduce to

$$\sum \int \frac{\sigma^2 - \epsilon_i^2}{\sigma^4} d\epsilon_i = \frac{\sigma^2 \sum \epsilon_i}{\sigma^4} - \frac{\sum \epsilon_i^3}{3\sigma^4}.$$

When we put $\theta = \tilde{\theta}$, $\sum \tilde{\epsilon}_i = 0$. So the first term is $n\mu_3/3\mu_2^2$. The second part is

$$-n\frac{\int \exp\left(\int \frac{c_1-\epsilon_i}{\sigma^2-c_1\epsilon_i}d\epsilon_i\right)\left[\int \frac{(\sigma^2-c_1\epsilon_i)-(c_1-\epsilon_i)(-\epsilon_i)}{(\sigma^2-c_1\epsilon_i)^2}d\epsilon_i\right]d\epsilon_i}{\int \psi(\epsilon_i)d\epsilon_i} = 0 \quad \text{at } c_1 = 0.$$

Therefore,

$$\left.\frac{\partial l(\theta)}{\partial \theta}\right|_{c_1=0} = \frac{n\mu_3}{3\mu_2^2}.$$

c. Testing $c_1 = 0$ is essentially checking the third moment $\mu_3 \simeq 0$. Therefore, it will be test for skewness. Since $H_0 : c_1 = 0$, we test only one restriction and the test statistic $\rightsquigarrow \chi_1^2$ under H_0.

6. [5+10]

a. Obvious.

b. $H_0 : \hat{\beta}_{pnc} = \hat{\beta}_{puc}$ against $H_a : \hat{\beta}_{pnc} \neq \hat{\beta}_{puc}$. The t-statistic (which follows $\sim t_{17}$ under H_0).

$$t = \frac{\hat{\beta}_{pnc} - \hat{\beta}_{puc}}{\sqrt{V(\hat{\beta}_{pnc}) + V(\hat{\beta}_{puc}) - 2COV(\hat{\beta}_{pnc}, \hat{\beta}_{puc})}} \simeq 1.08$$

Therefore, we cannot reject the null that the consumer don't differentiate between the changes in the prices of new and used car.

7. [10]

Consider the model with $k = 2$, $y = \beta_1 x_1 + \beta_2 x_2 + u$, where $\{u_i\}$ are i.i.d. mean 0 and variance σ^2 random variable. Assume also $\lim n^{-1}X'X = \mathbf{A}$, where $\mathbf{A}$ is a 2×2 nonsingular matrix. Then, $\hat{\beta}_1/\hat{\beta}_2$, where $\hat{\beta}_1$ and $\hat{\beta}_2$ are the least squares, assuming $\beta_2 \neq 0$, is asymptotically normal with variance-covariance matrix $\sigma^2\gamma'\mathbf{A}\gamma$, where $\gamma' = (\beta_2^{-1}, -\beta_1\beta_2^{-1})$.

Bonus question [10]:

The t-test for testing $H_0 : \beta_j = 0$ is given by $t = \hat{\beta}_j/(\text{s.e.}\hat{\beta}_j) = \hat{\beta}_j/([\hat{\sigma}^2(X'X)_{jj}^{-1}]^{1/2})$. Suppose under $H_a : \beta_j = \delta \neq 0$.

Then under H_a, plim $t = (\text{plim}\, \hat{\beta}_j \lim \sqrt{n})/(\text{plim}\, \hat{\sigma}^2 n\{\lim(X'X/n)^{-1}_{jj}\}^{1/2})$
$= \delta \lim \sqrt{n}/(\sigma\{\lim(X'X/n)^{-1}_{jj}\}^{1/2})$. Since we assumed $\lim(X'X/n) = Q$ finite, plim $r = \infty$. Since the test is unbounded as $n \to \infty$, we will reject H_0 with prob 1. We need other assumption that $\epsilon_i \sim IIDN(0, \sigma^2)$.