University of Illinois **Dept. of Economics**

Spring 2010 **Economics 507** **Sung Y. Park**

University of Illinois
Department of Economics

First Examination
Tuesday, March 30, 2010
19:00 - 22:00 pm, 120 Architecture Bldg.

Name: ____

ID# : ____

SOLUTION

This solution contains only answers. Your exam answers will be graded based on this solution, and they will be expected, at least, to have points along with the presented solution.

This is a closed-book examination. Answer as many questions as you can. Credit for each question is given in [ ]. GOOD LUCK!

1.
a. Note that: OLSE from (1) is $\hat{\beta}_1 = (X_1'M_2X_1)^{-1}X_1'M_2Y$, and OLSE from (2) is $\check{\beta}_1 = (X_1'X_2)^{-1}X_1'Y$.

Unbiasedness:

$\mathbb{E}(\hat{\beta}_1) = (X_1'M_2X_1)^{-1}X_1'M_2\mathbb{E}(Y) = (X_1'M_2X_1)^{-1}X_1'M_2X_1\beta_1 + (X_1'M_2X_1)X_1'M_2X_2\beta_2$ $= \beta_1$ (since $M_2X_2 = 0$). Therefore, $\hat{\beta}_1$ is always unbiased. Now, $\mathbb{E}(\check{\beta}_1) = (X_1'X_1)^{-1}X_1'$ $(X_1\beta_1 + X_2\beta_2) = \beta_1 + (X_1'X_1)^{-1}X_1'X_2\beta_2$. Therefore, $\mathbb{E}(\check{\beta}_1) = \beta_1$ only if $\beta_2 = 0$ (i.e., models (1) and (2) are the same) and/or $X_1'X_2 = 0$ (i.e., X_1 and X_2 matrices are orthogonal).

Efficiency:

We can write $\hat{\beta}_1 = \beta_1 + (X_1'M_2X_1)^{-1}X_1'M_2\epsilon$. Then, $V(\beta_1) = (X_1'M_2X_1)^{-1}X_1'M_2V(\epsilon)$ $M_2X_1'(X_1, M_2X_1)^{-1} = \sigma^2(X_1'M_2X_1)^{-1}$. We also have $V(\check{(\beta)}_1) = \sigma^2(X_1'X_1)^{-1}$. We have $V(\check{\beta}_1)^{-1} - V(\hat{\beta}_1)^{-1} = (1/\sigma^2)(X_1'X_2(X_2'X_2)^{-1}X_2'X_1)$ (p.s.d), and $V(\hat{\beta}_1) - V(\check{\beta}_1)$ is p.s.d. Therefore the $\check{\beta}_1$ is *always* more efficient than the $\hat{\beta}_1$ (whether or nor $\beta_2 = 0$).

b. We can avoid the use of partitioned inverse formula. Let us denote $X = [X_1, X_2]$ and $\beta = (\beta_1, \beta_2)'$. Then (1) is simply, $Y = X\beta + \epsilon$. Then $\hat{\beta}_{OLS} = (\hat{\beta}_1, \hat{\beta}_2)' = (X'X)^{-1}X'Y$. Now, $\mathbb{E}(\hat{\beta}_{OLS}) = (X'X)^{-1}X'\mathbb{E}(Y) = (X'X)^{-1}X'X_1\beta_1 = (X'X)^{-1}X'$ $(X_1, X_2)(\beta_1, 0)' = (\beta_1, 0)'$. Therefore, $\mathbb{E}(\hat{\beta}_1) = \beta_1$ and $\mathbb{E}(\hat{\beta}_2) = 0$. Hence, the OLS estimator of β_1 from the model (1) is unbiased.

2.
The constrained OLS estimator ia a solution to

$$\min_{\beta}(Y - X\beta)'(Y - X\beta) \quad \text{subject to} \quad R\beta = \delta.$$

The objective function is convex in β and the constraints are linear. Denoting the Lagrangian multipliers by λ the Lagrangian function is

$$\mathcal{L} = (Y - X\beta)'(Y - X\beta) - \lambda'(R\beta - \delta).$$

The FOCs are $-2X'(Y - X\hat{\beta}_n^0) - R'\hat{\lambda}_n = 0$ and $R\hat{\beta}_n^0 = \delta$. From the former equation we can get $\hat{\beta}_n^0 = \hat{\beta}_n + 1/2(X'X)^{-1}R'\hat{\lambda}_n$. And, this yields $\delta = R\hat{\beta}_n^0 = R\hat{\beta}_n + 1/2R(X'X)^{-1}R'\hat{\lambda}_n$. So we have $\hat{\lambda}_n = 2(R(X'X)^{-1}R')(\delta - R\hat{\beta}_n^0)$. Therefore, the constrained OLS estimator is

$$\hat{\beta}_n^0 = \hat{\beta}_n + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(\delta - R\hat{\beta}_n).$$

When the constraints are misspecified, $R\beta_0 \neq \delta$, the constrained OLS estimator converges to

$$\lim_{n\to\infty} \hat{\beta}_n^0 = \beta_0 + \mathbb{E}(X'X)^{-1}R'(R(\mathbb{E}X'X)^{-1}R')^{-1}(\delta - R\beta_0).$$

In general, Constrained OLS estimator is inconsistent. For $c \in R^p$ $c'\hat{\beta}_n^0$ is consistent for $c'\beta_0$ iff $c'\mathbb{E}(X'X)^{-1}R' = 0$.

3.

In this case, $h(\hat{\theta}) = \hat{\beta}_2$, and $H(\hat{\theta})$ is $k_2 \times k$ matrix $[\mathbf{0} \vdots \mathbf{I}]$, where $\mathbf{0_{k_2\times k_1}}$ and $\mathbf{I_{k_2\times k_2}}$. Then $W = \hat{\beta}_2'\hat{\mathbb{V}}(\hat{\beta}_2)^{-1}\hat{\beta}_2$. Since $\hat{\beta}_2 = (X_2'M_1X_2)^{-1}X_2'M_1y$ and $\hat{\mathbb{V}}(\hat{\beta}_2) = \hat{\sigma}^2(X_2'M_1X_2)^{-1}$ the Wald statistic is

$$W = \frac{1}{\hat{\sigma}^2}y'M_1X_2(X_2'M_1X_2)^{-1}X_2'M_1y.$$

The conventional F test for $\beta_2 = 0$ is

$$F = \frac{n-k}{k_2} \times \frac{y'M_1X_2(X_2'M_1X_2)^{-1}X_2'M_1y}{y'M_Xy}.$$

Thus the Wald statistic is nothing but $y'M_1X_2(X_2'M_1X_2)^{-1}X_2'M_1y/(y'M_Xy/n)$. Thus

$$W = \frac{nk_2}{n-k}F.$$

4.

When θ_3 and θ_4 are equal to zero, $f_1(x)$ is nothing but the standard normal probability density function. Moreover, since the log density is expressed in a very simple form: $\log f_1(x) = -\sum_{k=0}^{4}\theta_k x^k$, the score function under the null is $\hat{S}_1 = n[0,0,0,\hat{\mu}_2,\hat{\mu}_4 - 3]$. Thus the LM test statistics is $(1/n)\hat{S}_1'\mathcal{J}_1^{-1}\hat{S}_1 = n(\hat{\mu}_3^2/6 + (\hat{\mu}_4^2 - 3)^2/24)$. The LM test statistics is exactly the same as Jarque and Bera's normality test. Since skewness of the distribution of financial asset returns is assumed to be zero this test is hard to be applied in this case because the quartic exponential distribution does not admit the symmetric and leptokutic behavior.

6.

We know that $\mathbb{V}(\hat{\beta}_{OLS}) = (X'X)^{-1}X'\Sigma X(X'X)^{-1}$ where $\Sigma = \mathbb{V}(\epsilon)$. After simple calculation we obtain

$$(X'X)^{-1}X'\Sigma X(X'X)^{-1} = \begin{bmatrix} 0.435 & -0.073 \\ -0.073 & 0.0197 \end{bmatrix}.$$

Thus the correct standard errors for $\hat{\beta}_{1,ols}$ and $\hat{\beta}_{2,ols}$ are 0.66 and 0.14, respectively. So we can say that the conventional formula ($\sigma^2(X'X)^{-1}$) gives lower s.e. than the true variance formula.

7.
Consider the linear regression model with heteroskedasticity as $y = X\beta + \epsilon$, where $\mathbb{V}(\epsilon) = \Sigma = \text{diag}(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)$. We have $\hat{\beta}_{OLS} = (X'X)^{-1}X'y$. Hence the true variance $\mathbb{V}(\hat{\beta}_{OLS}) = (X'X)^{-1}X'\Sigma X(X'X)^{-1}$. White considered an estimator of Σ as $\hat{\Sigma} = \text{diag}(\hat{\sigma}_1^2, \hat{\sigma}_2^2, \cdots, \hat{\sigma}_n^2)$, where $\hat{\epsilon} = y - X\hat{\beta}$ and showed $\text{plim}(1/n)X'\hat{\Sigma}X = \lim(1/n)X'\Sigma X$. Hence a consistent estimator of $\mathbb{V}(\hat{\beta}_{OLS})$ is $(X'X)^{-1}X'\hat{\Sigma}X(X'X)^{-1}$.

8.
This is a question about simple calculation of conditional mean and variance. Obvious. The derived conditional variance function is the same as the ARCH(q) model. This implies that the consideration of randomness in conditional mean parameters affect conditional variation of the variable.

Bonus. [You should write down your derivation explicitly]
The correlation between $\hat{\beta}_1$ and $\hat{\beta}_2$ is $\mathbb{C}ov(\hat{\beta}_1, \hat{\beta}_2)/(\mathbb{V}(\hat{\beta}_1)\mathbb{V}(\hat{\beta}_2))^{1/2}$ conditional on x_1 and x_2. Since $\mathbb{V}(\hat{\beta}_1) = \sigma^2(x_1'M_2x_1)^{-1}$ and $\mathbb{V}(\hat{\beta}_2) = \sigma^2(x_2'M_1x_2)^{-1}$ we can write $x_1'M_2x_1$ as $(1-((x_1'x_2)^2)/((x_1'x_1)(x_2'x_2))x_1'x_1$. Thus $x_1'M_2x_1 = (1-\hat{\rho}^2)x_1'x_1$. The same applies to $x_2'M_1x_2$. Thus $\mathbb{V}(\hat{\beta}_1) = \sigma^2(1-\hat{\rho}^2)x_1'x_1$ and $\mathbb{V}(\hat{\beta}_2) = \sigma^2(1-\hat{\rho}^2)x_2'x_2$. By calculation one can show that $\mathbb{C}ov(\hat{\beta}_1, \hat{\beta}_2) = \sigma^2(x_1'M_2x_1)^{-1}x_1'M_2M_1x_2(x_2'M_1x_2)^{-1}$. Here $x_1'M_2M_1x_2 = (\hat{\rho}^2 - 1)x_1'x_2$. Thus we can write

$$\mathbb{C}ov(\hat{\beta}_1, \hat{\beta}_2) = \frac{(\hat{\rho}^2 - 1)x_1'x_2}{\sigma^2 x_1'x_1x_2'x_2}.$$

This finally yields

$$\frac{\mathbb{C}ov(\hat{\beta}_1, \hat{\beta}_2)}{(\mathbb{V}(\hat{\beta}_1)\mathbb{V}(\hat{\beta}_2))^{1/2}} = \frac{(\hat{\rho}^2 - 1)x_1'x_2}{(1-\hat{\rho}^2)(x_1'x_1)^{1/2}(x_2'x_2)^{1/2}} = -\hat{\rho}.$$