Xiamen University **WISE**

Spring 2008 **Econometrics II** **Sung Yong Park**

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The Wang Yanan Institute for Studies in Economics

Final Examination
Friday, June 13, 2008
2:00 - 5:00 pm, 207 Jia-Geng IV.

Name: ______________________________

ID# : ______________________________

SOLUTION

This solution contains only answers. Your exam answers will be graded based on this solution, and they will be expected, at least, to have points along with the presented solution.

This is a closed-book examination. Answer as many questions as you can. Credit for each question is given in [ ]. GOOD LUCK!

1.

This is a case of heteroskedasticity. The most efficient estimator of β is GLS estimator.

$$\hat{\beta}_{GLS} = (X'\Sigma X)^{-1}X'\Sigma Y,$$

where

$$\Sigma = \sigma^2 \begin{bmatrix} x_1^2 & & & \\ & x_2^2 & & \\ & & .. & \\ & & & x_n^2 \end{bmatrix}$$

Thus,

$$\hat{\beta}_{GLS} = \frac{1}{n}\sum y_i/x_i.$$

$$Var(\hat{\beta}_{GLS}) = V((1/n)\sum(y_i/x_i)) = V((1/n)\sum(\beta x_i + \epsilon_i)/x_i)$$
$$= (1/n^2)\sum V(\epsilon_i)/x_i^2 = \sigma^2/n$$

We know that the OLS estimator is

$$\hat{\beta}_{OLS} = \frac{\sum y_i x_i}{\sum x_i^2} = \beta + \frac{\sum y_i \epsilon_i}{\sum x_i^2}$$

Hence,

$$V(\hat{\beta}_{OLS}) = \sigma^2 \frac{\sum x_i^4}{(\sum x_i^2)^2}$$

Now

$$\frac{V(\hat{\beta}_{OLS})}{V(\hat{\beta}_{GLS})} = \frac{n\sum x_i^2}{(\sum x_i^2)^2}$$

By Cauchy inequality $\sum x_i^2 n \geqslant (\sum x_i^2)^2 \Rightarrow V(\hat{\beta}_{OLS}) \geqslant V(\hat{\beta}_{GLS})$.

2. [15]

Since u_t's are $N(0, \sigma^2 + Z_t'\alpha)$ and independent, we can write

$$f(u_1, \cdots, u_T) = \prod_{t=1}^{T}\left(\frac{1}{\sqrt{2\pi(\sigma^2 + Z_t'\alpha)}}\exp\left[-\frac{u_t^2}{2(\sigma^2 Z_t'\alpha)}\right]\right).$$

Thus,

$$\log f(u_1, \cdots, u_T) = -\frac{T}{2}\log 2\pi - \frac{1}{2}\sum \log\sigma_t^2 - \sum\frac{u_t^2}{\sigma_t^2},$$

where $\sigma_t^2 = \sigma^2 + Z_t'\alpha$. Now $u_t = \epsilon_t - \rho\epsilon_{t-1}$

$$\log f(\epsilon_1, \cdots, \epsilon_T) = -\frac{T}{2}\log 2\pi - \frac{1}{2}\sum \log \sigma_t^2 - \sum \frac{(\epsilon_t - \rho\epsilon_{t-1})^2}{\sigma_t^2},$$

But, $\epsilon_t = y_t - x_t'\beta$. Hence the log likelihood function in terms of $y_1, \cdots, y_T$ can be written as

$$l(\theta) = -\frac{T}{2}\log 2\pi - \frac{1}{2}\sum \log \sigma_t^2 - \sum \frac{[(y_t - x_t'\beta) - \rho(y_{t-1}x_t'\beta)]^2}{\sigma_t^2},$$

where $\theta = (\beta', \sigma^2, \alpha', \rho)'$.

To test the autocorrelation and heteroscedasticity together, we can consider a LM-test given the null:

$$H_0 : \rho = \alpha_1 = \alpha_2 = ... = \alpha_p = 0$$

The score vector is:

$$\left.\frac{\partial l(\theta)}{\partial \alpha}\right|_{H_0} = \frac{1}{2\sigma^4}\sum \epsilon_t^2 Z_t,$$

$$\left.\frac{\partial l(\theta)}{\partial \rho}\right|_{H_0} = \frac{1}{\sigma^2}\sum \epsilon_t\epsilon_{t-1}$$

A joint test will be based on the correlation between (ϵ_t^2 and Z_t) and (ϵ_t and ϵ_{t-1}).

3.

Rewrite the model as

$$y_t = x_t'\beta + 0.5(y_{t-1} - x_{t-1}'\beta) + u_t$$

$$y_t - 0.5y_{t-1} = (x_t - 0.5x_{t-1})'\beta + u_t$$

with $u_t \sim i.i.d(0, \sigma_u^2)$. Let

$$\begin{aligned} y_t^* &= y_t - 0.5y_{t-1} \\ x_t^* &= x_t - 0.5x_{t-1} \end{aligned}$$

We have

$$y_t^* = x_t^{*\prime}\beta + u_t.$$

Then, we get $\hat{\beta}_{OLS} = (X^{*\prime}X^*)^{-1}X^*Y$.

Since, here we have a lagged dependent variable in the model we have to apply Durbin-h test.

$$h = \frac{\hat{\rho}}{\sqrt{\frac{1}{n} - se^2(\phi_1)}} \rightarrow N(0,1)$$

Where ϕ_1 is the AR(1) coefficient, and $n = 21$.

$$\begin{aligned} \hat{\rho} &= 1 - \frac{DW}{2} = 0.395 \\ se(\phi_1) &= \frac{\hat{\phi}_1}{t} = \frac{0.97}{5.39} = 0.18 \end{aligned}$$

Then,

$$h = \frac{0.395}{\sqrt{\frac{1}{21} - 0.18^2}} = 3.2 > Z_{0.01}$$

Therefore the null hypothesis is rejected and the disturbances are autocorrelated.

4

By $\varepsilon_t|\psi_t \sim N(0, \sigma_t^2)$, We have the conditional variance and kurtosis:

$$\begin{aligned} E(\varepsilon_t^2|\psi_t) &= \sigma_t^2 \\ E(\varepsilon_t^4|\psi_t) &= 3\sigma_t^4 \end{aligned}$$

Therefore,

$$\begin{aligned} E(\varepsilon_t^2) &= E(E(\varepsilon_t^2|\psi_t)) = E(\sigma_t^2) \\ E(\varepsilon_t^4) &= E(E(\varepsilon_t^4|\psi_t)) = 3E(\sigma_t^4) \end{aligned}$$

Since we have

$$\begin{aligned} Var(\varepsilon_t) &= E(Var(\varepsilon_t|\psi_t)) + Var(E(\varepsilon_t|\psi_t)) \\ &= E(\sigma_t^4) - (E(\sigma_t^2))^2 > 0 \end{aligned}$$

$$E(\sigma_t^4) > (E(\sigma_t^2))^2$$

And, therefore,

$$K_{uc} = \frac{E(\varepsilon_t^4)}{(E(\varepsilon_t^2))^2} = \frac{3E(\sigma_t^4)}{(E(\sigma_t^2))^2} > \frac{3E(\sigma_t^4)}{E(\sigma_t^4)} = 3$$

Hence, unconditional ε_t's are not normal.

Note that by adding only one extra parameter we improve the log-likelihood function by 25. LM test for normality of the ARCH model is also lower, though still quite significant. The ARCH coefficient is significant. The t-stat for the regression coefficients are also higher for the ARCH-regression model. Since ARCH-regression model capture the conditional heteroskedasticity, these estimators are more efficient. Therefore, aside from complexity, the ARCH-regression model is better then OLS model in all aspect.

5.

a)
[2SLS]

$$\begin{aligned}\hat{\beta}_{2SLS} &= (\hat{X}'\hat{X})^{-1}\hat{X}Y \\ &= (X'Z(Z'Z)^{-1}Z'Z(Z'Z)^{-1}Z'X)^{-1}XZ(Z'Z)ZY \\ &= (X'P_ZX)^{-1}XP_ZY\end{aligned}$$

[IV]

$$\begin{aligned}\hat{\beta}_{IV} &= (\hat{X}'X)^{-1}\hat{X}Y \\ &= (X'Z(Z'Z)^{-1}Z'X)^{-1}XZ(Z'Z)Z'Y \\ &= (X'P_ZX)^{-1}XP_ZY\end{aligned}$$

$$\hat{\beta}_{2SLS} = \hat{\beta}_{IV}$$

b)
VC (Variance-covariance) matrix of 2SLS estimator $\tilde{\beta}$ is $Var(\tilde{\beta} = \sigma_u^2(\hat{X}'\hat{X})^{-1} \rightarrow \hat{Var}(\tilde{\beta} = \hat{\sigma}_u^2(\hat{X}'\hat{X})^{-1}$, where $\hat{\sigma}_u^2 = \hat{u}'\hat{u}/(n-p) = (y - X\tilde{\beta})'(y - X\tilde{\beta})/(n-p)$. VC matrix of the OLS estimator $\hat{\beta}$ is $Var(\hat{\beta} = \sigma_v^2(\hat{X}'\hat{X})^{-1} \rightarrow \hat{Var}(\hat{\beta} = \hat{\sigma}_v^2(\hat{X}'\hat{X})^{-1}$, where $\hat{\sigma}_v^2 = \hat{v}'\hat{v}/(n-p) = (y - \hat{X}\hat{\beta})'(y - \hat{X}\hat{\beta})/(n-p)$. Since $\hat{X} \neq X$, $\hat{\sigma}_v^2 \neq \hat{\sigma}_u^2$ though $\tilde{\beta} = \hat{\beta}$. So OLS estimation gives us a wrong VC matrix for $\tilde{\beta}$ which is a 2SLS estimator.

6

a) Since we have

$$\begin{aligned}\text{logit}(P_i) &= -4.5 + 1.7x_i - 0.25x_i^2 \\ \ln(\frac{P}{1-P}) &= -4.5 + 1.7x_i - 0.25x_i^2\end{aligned}$$

FOC:

$$\begin{aligned}1.7 - 0.5x_i &= 0 \\ x_i &= 3.4\end{aligned}$$

Therefore the optimal number of pages is

$$n = e^{3.4} \approx 30$$

b)

$$\begin{aligned}\ln(\frac{P_1}{1-P_1}) &= -4.5 + 1.7\ln 40 - 0.25(\ln 40)^2 = -1.6309 \\ \ln(\frac{P_2}{1-P_2}) &= -4.5 + 1.7\ln 50 - 0.25(\ln 50)^2 = -1.6755\end{aligned}$$

Then, we have

$$\begin{aligned}P_1 &= \frac{e^{-1.6309}}{1+e^{-1.6309}} = 0.16371 \\ P_2 &= \frac{e^{-1.6755}}{1+e^{-1.6755}} = 0.15769\end{aligned}$$

So the change of the probability is $P_1 - P_2 = 0.16371 - 0.15769 = 0.00602$.

7

[GMM]: Given the moment conditions:

$$E(x_i\varepsilon_i) = E(x_i(y_i - x_i'\beta)) = g(x_i, \theta)$$

We consider the GMM estimator:

$$\hat{\theta} = \arg\min_{\theta}(\frac{1}{n}\sum g(x_i,\theta)W^{-1}\frac{1}{n}\sum g(x_i,\theta))$$

Where W is a consistent estimator of $E(g(x_i,\theta)'g(x_i,\theta))$. To implement the GMME, we need to estimate this weight matrix first (2-stage GMM).
[EL] Empirical Likelihood estimator is defined as:

$$\begin{aligned}\hat{\theta} &= \arg\max_{\theta, P_1 \ldots P_n}\sum \ln P_i \\ s.t &: \sum P_i g(x_i,\theta) = 0, \sum P_i = 1, P_i > 0\end{aligned}$$

[ET] The Exponential Tilting (ET) estimator is defined as:

$$\hat{\theta} = \arg\min_{\theta, P_1 \ldots P_n}(-\sum P_i \ln P_i)$$
$$s.t \ : \sum P_i g(x_i,\theta) = 0, \sum P_i = 1, P_i > 0$$

All of these three estimators are consistent, asymptotically normal. Small sample properties of these estimators, however, might be different each other: ET and EL estimators have smaller bias than GMM estimator.

8

Note that

$$P(Y_i = y_i) = \frac{e^{-\beta x_i}(\beta x_i)^{y_i}}{y_i!}$$

$$\begin{aligned} l(\theta) &= \sum \ln f(y_i) = \sum[-\beta x_i + y_i \ln \beta + y_i \ln x_i - \ln(y_i!)] \\ &= -\beta \sum x_i + \ln \beta \sum y_i + \sum y_i \ln x_i - \sum \ln(y_i!) \end{aligned}$$

$$\begin{aligned} \left.\frac{\partial l(\theta)}{\partial \beta}\right|_{\hat{\beta}} = 0 \;\; &\Rightarrow \;\; -\sum x_i + \frac{1}{\hat{\beta}} \sum y_i = 0 \\ \hat{\beta}_{MLE} &= \frac{\sum y_i}{\sum x_i} \end{aligned}$$

To show $\hat{\beta}$ is consistent we only check $\lim_{n\to\infty} E(\hat{\beta}) = \beta$ and $\lim_{n\to\infty} Var(\hat{\beta}) = 0$. Note that $E(y_i) = \beta x_i$ and $Var(y_i) = \beta x_i$ (since $y_i \sim$ Poisson)

$$\begin{aligned} E(\hat{\beta}) &= \frac{E(\sum y_i)}{\sum x_i} = \frac{\beta \sum x_i}{\sum x_i} = \beta \\ Var(\hat{\beta}) &= Var\left(\frac{\sum y_i}{\sum x_i}\right) = \frac{\beta \sum x_i}{(\sum x_i)^2} = \frac{\beta}{\sum x_i} \to 0 \text{ as n} \to \infty \end{aligned}$$

Therefore $\hat{\beta}_{MLE}$ is consistent.