ECON 2121B
Methods of Economic Statistics

Chapter 6
Continuous Probability Distributions

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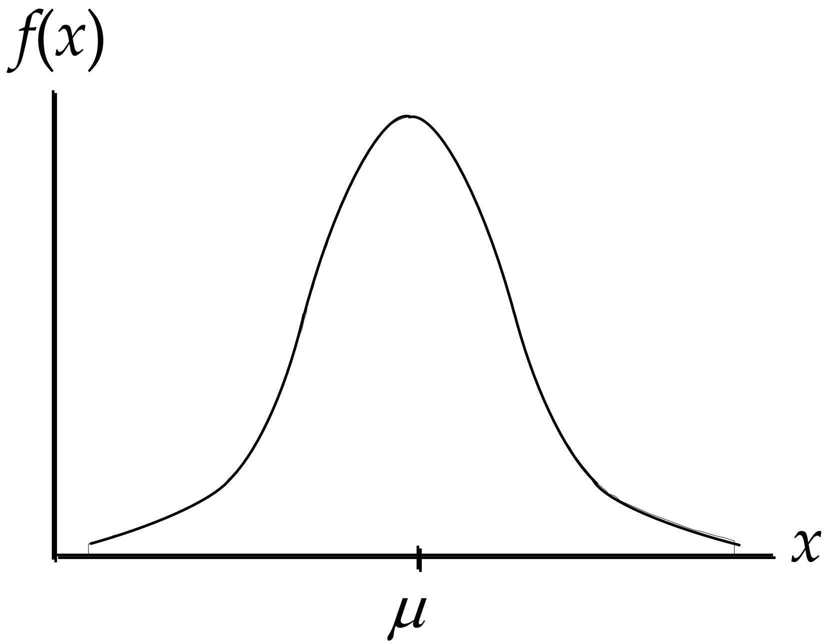
Continuous Probability Distributions

- Uniform Probability Distribution
- Normal Probability Distribution
- Exponential Probability Distribution



Continuous Probability Distributions

- A continuous random variable can assume any value in an interval on the real line or in a collection of intervals.
- It is not possible to talk about the probability of the continuous random variable assuming a particular value.
- Instead, we talk about the probability of the continuous random variable assuming a value within a given interval.
- The probability of the continuous random variable assuming a value within some given interval from x_1 to x_2 is defined to be the area under the graph of the probability density function between x_1 and x_2.

Uniform Probability Distribution

- A random variable is uniformly distributed whenever the probability is proportional to the interval's length.
- Uniform Probability Density Function

$$f(x) = 1/(b - a) \quad \text{for } a \leq x \leq b$$
$$= 0 \quad \text{elsewhere}$$

where: a = smallest value the variable can assume

b = largest value the variable can assume

Uniform Probability Distribution

- Expected Value of x

$$E(x) = (a + b)/2$$

- Variance of x

$$Var(x) = (b - a)^2/12$$

where: a = smallest value the variable can assume
b = largest value the variable can assume

$$\mu = \int_a^b x f(x) dx = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x dx$$

$$= \frac{1}{b-a} (\frac{1}{2} x^2) |_a^b = \frac{1}{b-a} \frac{1}{2} (b^2 - a^2) = \frac{a+b}{2}$$

$$\sigma^2 = \int_a^b (x-\mu)^2 f(x) dx = \int_c^d (x - \frac{a+b}{2})^2 \frac{1}{b-a} dx$$

$$Denotes\ t = x - \frac{a+b}{2},\ \ a \le x \le b\ indicates\ \frac{a-b}{2} < t < \frac{b-a}{2}$$

$$Thus,$$

$$\sigma^2 = \frac{1}{b-a} \int_{(a-b)/2}^{(b-a)/2} t^2 dt = \frac{1}{b-a} \frac{1}{3} t^3 |_{(b-a)/2}^{(b-a)/2}$$

$$= \frac{1}{3(b-a)} [(\frac{b-a}{2})^3 - (\frac{b-a}{2})^3] = \frac{1}{3(b-a)} \frac{(b-a)^3}{4} = \frac{(b-a)^2}{12}$$

Example: Slater's Buffet

- Uniform Probability Distribution

  Slater customers are charged for the amount of salad they take. Sampling suggests that the amount of salad taken is uniformly distributed between 5 ounces and 15 ounces.

  The probability density function is

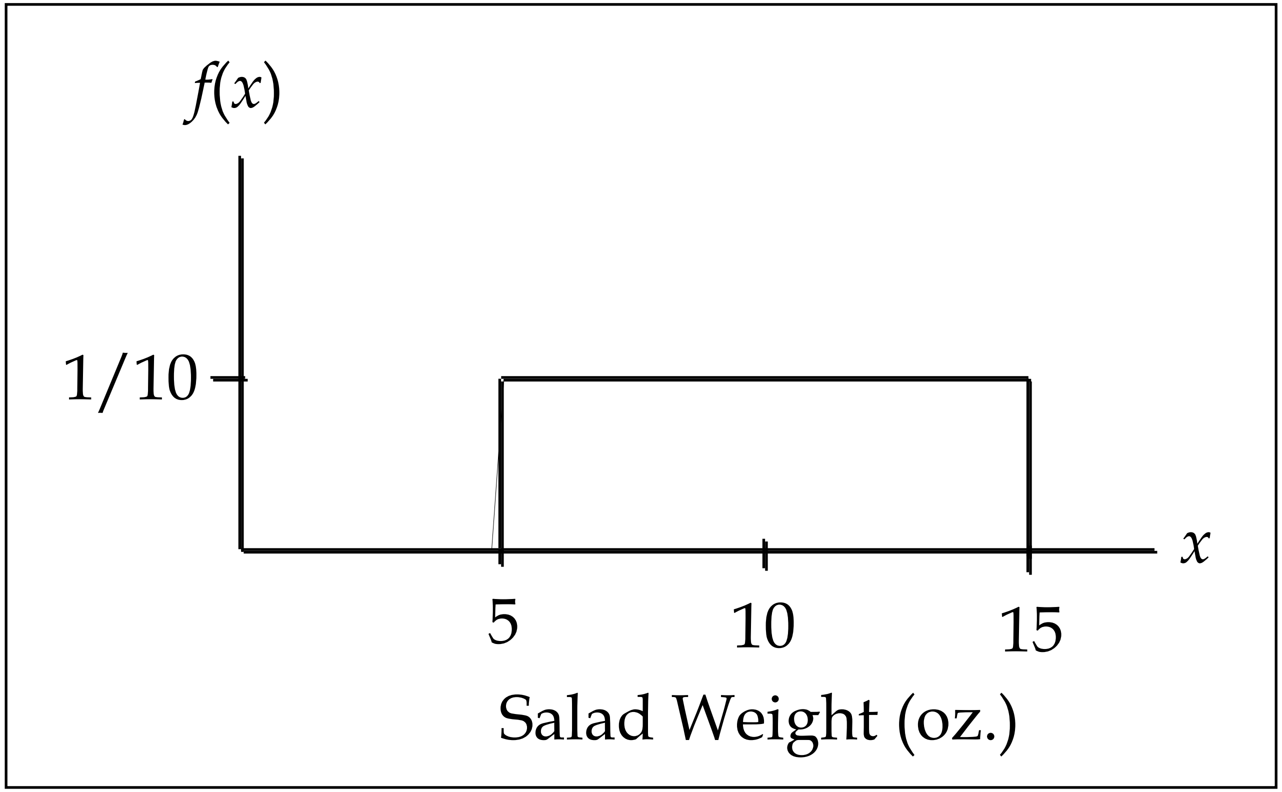
  $$f(x) = 1/10 \quad \text{for } 5 \leq x \leq 15$$
  $$= 0 \quad \text{elsewhere}$$

  where:

  x = salad plate filling weight

Example: Slater's Buffet

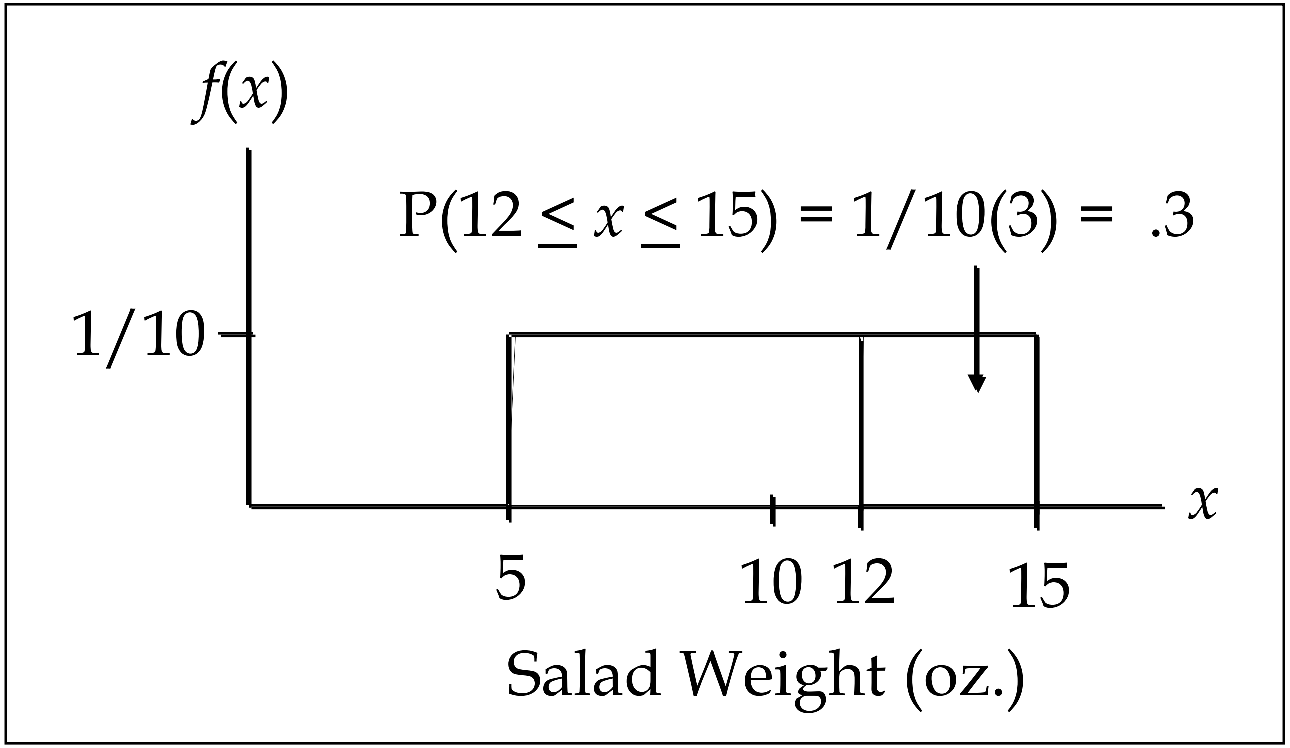
- Uniform Probability Distribution for Salad Plate Filling Weight



Example: Slater's Buffet

- Uniform Probability Distribution

  What is the probability that a customer will take between 12 and 15 ounces of salad?



Example: Slater's Buffet

- Expected Value of x

$$E(x) = (a + b)/2$$
$$= (5 + 15)/2$$
$$= 10$$

- Variance of x

$$Var(x) = (b - a)^2/12$$
$$= (15 – 5)^2/12$$
$$= 8.33$$

Normal Probability Distribution

- The normal probability distribution is the most important distribution for describing a continuous random variable.
- It has been used in a wide variety of applications:
  - Heights and weights of people
  - Test scores
  - Scientific measurements
  - Amounts of rainfall
- It is widely used in statistical inference

Normal Probability Distribution

- Normal Probability Density Function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

where:

μ = mean

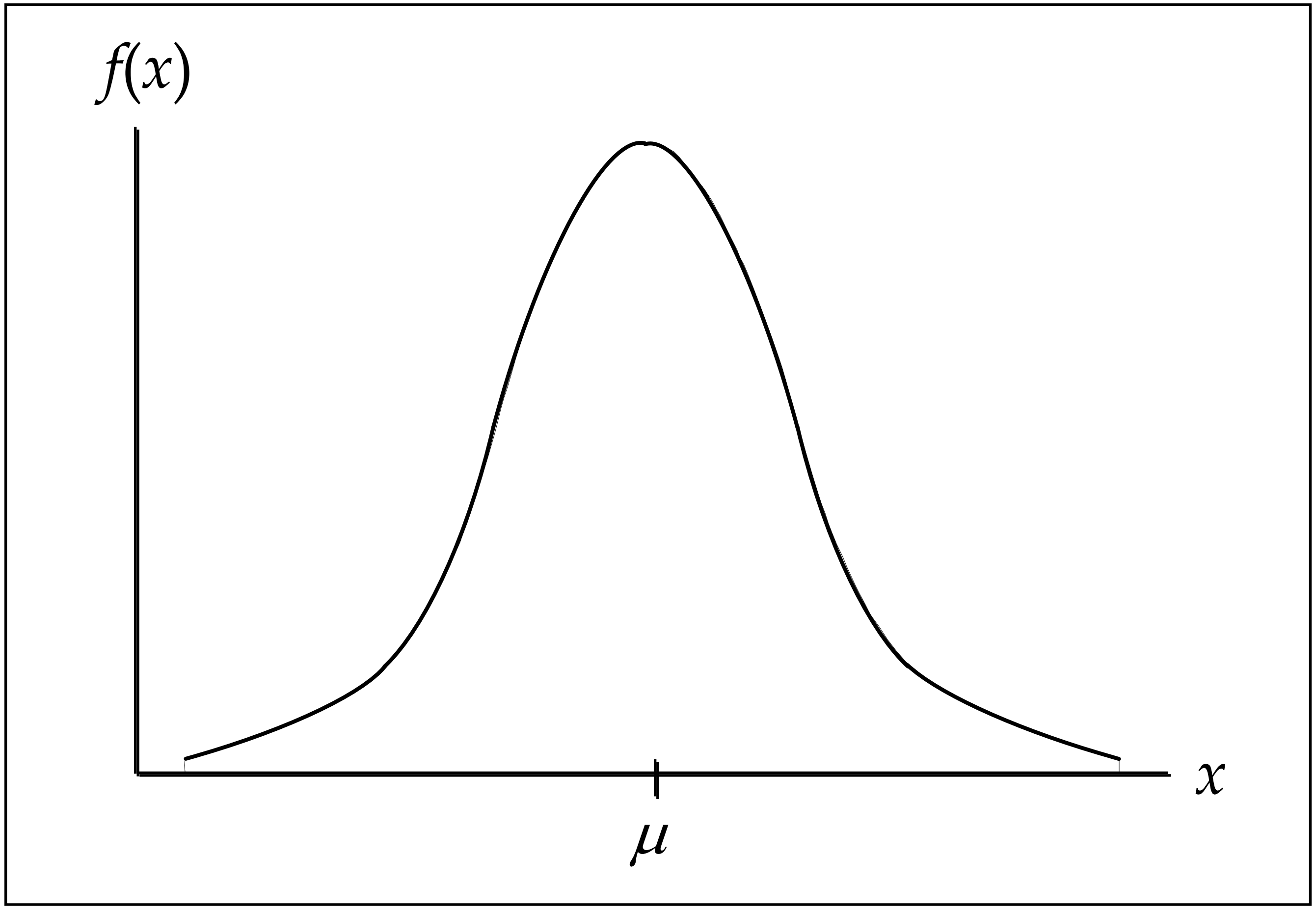
σ = standard deviation

π = 3.14159

e = 2.71828

Normal Probability Distribution

- Graph of the Normal Probability Density Function



Normal Probability Distribution

- Characteristics of the Normal Probability Distribution
  - The distribution is symmetric, and is often illustrated as a bell-shaped curve.
  - Two parameters, μ (mean) and σ (standard deviation), determine the location and shape of the distribution.
  - The highest point on the normal curve is at the mean, which is also the median and mode.
  - The mean can be any numerical value: negative, zero, or positive.

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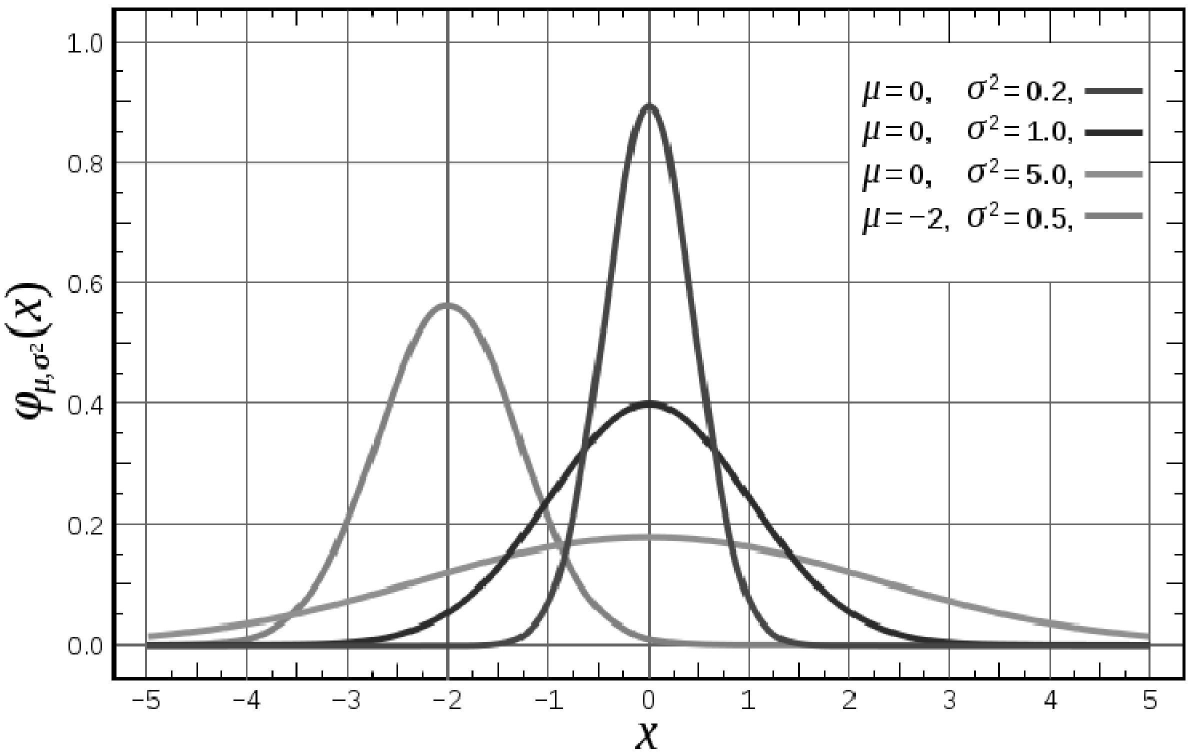
Characteristics of the Normal Probability Distribution

- Each combination of μ and **σ** produces a unique normal curve
- The standard deviation determines the width of the curve: larger values result in wider, flatter curves.



Normal Probability Distribution

- Characteristics of the Normal Probability Distribution
  - The total area under the curve is 1 (.5 to the left of the mean and .5 to the right).
  - Probabilities for the normal random variable are given by areas under the curve.

Normal Probability Distribution

- Characteristics of the Normal Probability Distribution
  - **68.26%** of values of a normal random variable are within **+/- 1 standard deviation** of its mean.
  - **95.44%** of values of a normal random variable are within **+/- 2 standard deviations** of its mean.
  - **99.72%** of values of a normal random variable are within **+/- 3 standard deviations** of its mean.

Standard Normal Probability Distribution

- A random variable that has a normal distribution with a mean of zero and a standard deviation of one is said to have a standard normal probability distribution, denoted by **N(0,1).**

- The letter z is commonly used to designate this normal random variable.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Z-Values

- For any normally distributed random variable

$$x \sim N(\mu, \sigma^2)$$

- We can convert x to its z values as the following

$$z_i = \frac{x_i - \mu}{\sigma}$$

- The z value would have a standard normal distribution

$$z \sim N(0,1)$$

So ***any*** normally distributed variable can be analyzed with the standard normal distribution.

Example: Pep Zone

- Standard Normal Probability Distribution

  Pep Zone sells auto parts and supplies including a popular multi-grade motor oil. When the stock of this oil drops to 20 gallons, a replenishment order is placed.

  The store manager is concerned that sales are being lost due to stockouts while waiting for an order. It has been determined that leadtime demand is normally distributed with a mean of 15 gallons and a standard deviation of 6 gallons.

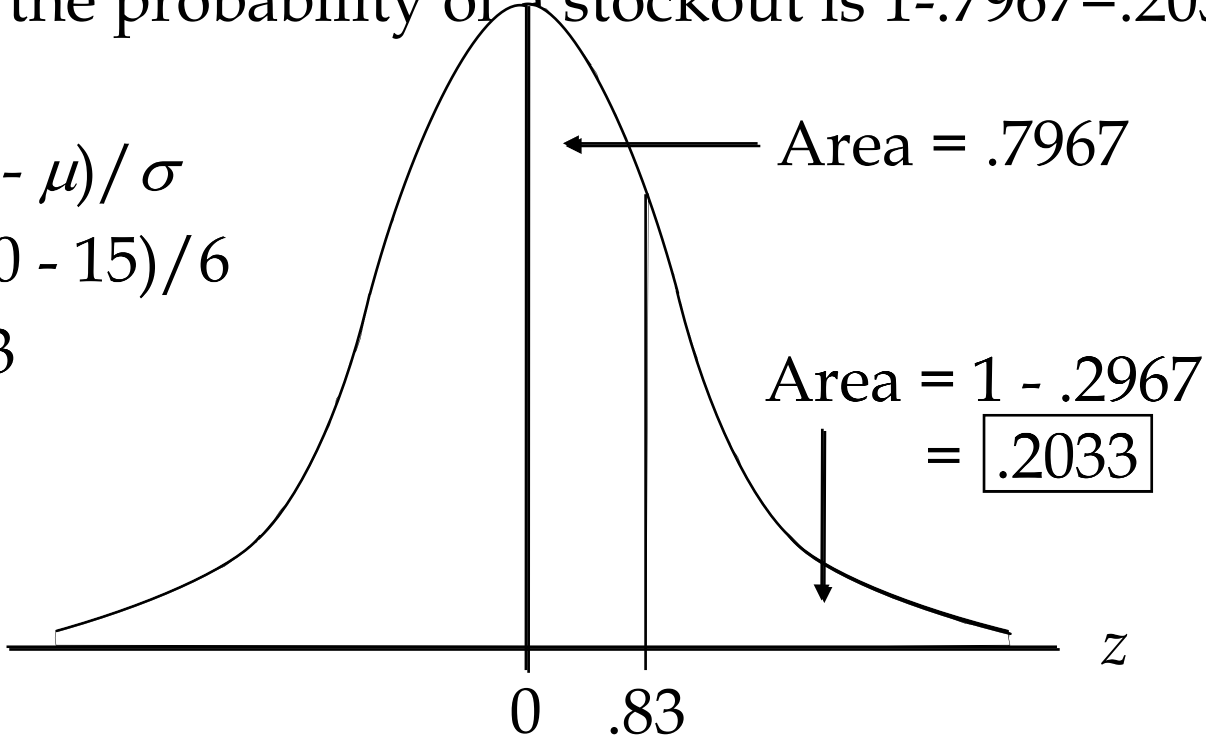
  The manager would like to know the probability of a stockout, P(x > 20).

Example: Pep Zone

- Standard Normal Probability Distribution

  The Cumulative Probabilities for Standard Normal Distribution Table shows an area of .7967 for the region under the curve to the left of the line z = .83. Thus, the probability of a stockout is 1-.7967=.2033.

$$z = (x - \mu)/\sigma$$
$$= (20 - 15)/6$$
$$= .83$$



Example: Pep Zone

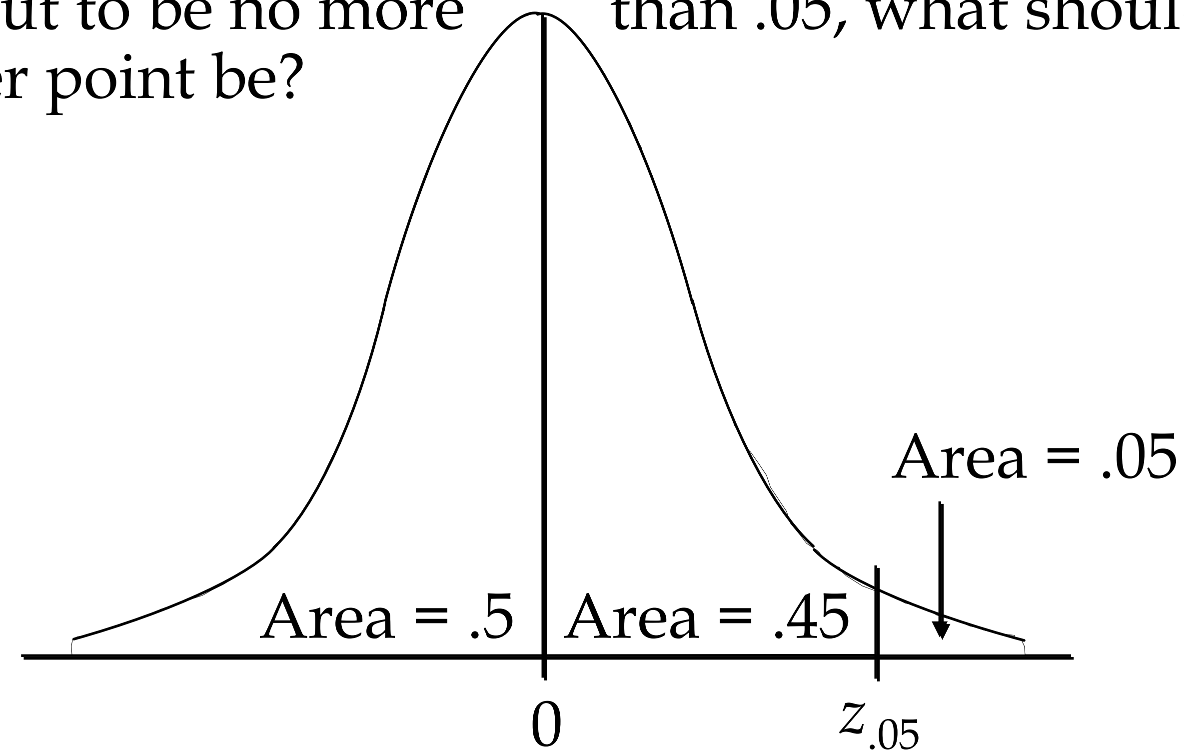
- Using the Standard Normal Probability Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7518	.7549
.7	.7580	.7612	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389

Example: Pep Zone

- Standard Normal Probability Distribution

  If the manager of Pep Zone wants the probability of a stockout to be no more than .05, what should the reorder point be?



Let $z_{.05}$ represent the z value cutting the .05 tail area.

Example: Pep Zone

- Using the Standard Normal Probability Table

  We now look-up the .9500 area in the Standard Normal Probability table to find the corresponding $z_{.05}$ value.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.	.	.	.	.	.	.	.	.	.	.
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
.	.	.	.	.	.	.	.	.	.	.

$z_{.05}$ = 1.645 is a reasonable estimate.

Example: Pep Zone

- Standard Normal Probability Distribution

  The corresponding value of x is given by

$$x = \mu + z_{.05}\sigma$$
$$= 15 + 1.645(6)$$
$$= 24.87$$

A reorder point of 24.87 gallons will place the probability of a stockout during leadtime at .05.

Perhaps Pep Zone should set the reorder point at 25 gallons to keep the probability under .05.

The Normal Distribution

- Say a toy car goes an average of 3,000 yards between recharges, with a standard deviation of 50 yards (i.e., $\mu = 3{,}000$ and $\sigma = 50$)
- What is the probability that the car will go more than 3,100 yards without recharging?

Example

- Say a toy car goes an average of 3,000 yards between recharges, with a standard deviation of 50 yards (i.e., $\mu = 3{,}000$ and $\sigma = 50$)
- What is the probability that the car will go more than 3,100 yards without recharging?

$$P(x > 3100) = P\left(z > \frac{3100 - 3000}{50} \right) =$$

$$P(z > 2.00) = 1 - P(z < 2.00)$$

$$= 1 - 0.9772 = 0.0228$$

Approximating a Binomial Distribution with the Normal Distribution

- Discrete calculations may become very cumbersome
- The normal distribution may be used to approximate discrete distributions
  - The larger *n* is, and the closer *p* is to .5, the better the approximation
- Since we need a range, not a value, in the normal distribution, the **correction for continuity** must be used
  - $P(x=r)=P(r-0.5<x<r+0.5)=P(x<r+0.5)-P(x<r-0.5)$

Approximating a Binomial Distribution with the Normal Distribution

Flip a coin 100 times and compare the binomial and normal results of getting 50% heads.

Binomial:

$$P(x=50)=\binom{100}{50}.5^{50}.5^{50}=.0796$$

Normal:

$$\mu=100\cdot.5=50$$

$$\sigma=\sqrt{100\cdot.5\cdot.5}=5$$

$$P(49.5<x<50.5)=P\left(\frac{49.5-50}{5}<z<\frac{50.5-50}{5}\right)=$$

$$P(-0.1<z<=0.1)=P(z<0.1)-P(z<-0.1)$$

$$=.5398-.4602=.0796$$

Approximating a Binomial Distribution with the Normal Distribution

Flip a weighted coin *[P(H)=.4]* 10 times and compare the results of getting 50% heads.

Binomial:	$P(x=5)=\binom{10}{5}.4^5.6^5=.2007$
Normal:	$\mu=10\cdot.4=4$ $\sigma=\sqrt{10\cdot.4\cdot.6}=1.55$ $P(4.5\le x\le 5.5)=P\left(\frac{4.5-4}{1.55}\le z\le\frac{5.5-4}{1.55}\right)=$ $P(0.32\le z\le 0.97)=.8340-.6179=.2085$

Approximating a Binomial Distribution with the Normal Distribution

Flip a weighted coin *[P(H)=.4]* 10 times and compare the results of getting 50% heads.

Binomial: $P(x=5)=\binom{10}{5}.4^5.6^5=.2007$

Normal: $\mu=10\cdot.4=4$

$\sigma=\sqrt{10\cdot.4\cdot.6}=1.55$

$P(4.5\le x\le 5.5)=P\left(\frac{4.5-4}{1.55}\le z\le\frac{5.5-4}{1.55}\right)=$

$P(0.32\le z\le 0.97)=.3340-.1179=.2085$

The more *p* differs from .5, and the smaller n is, the less precise the approximation will be.

Exponential Probability Distribution

- The exponential probability distribution is useful in describing the time it takes to complete a task.
- The exponential random variables can be used to describe:
  - Time between vehicle arrivals at a toll booth
  - Time required to complete a questionnaire
  - Distance between major defects in a highway

Exponential Probability Distribution

- Exponential Probability Density Function

$$f(x) = \frac{1}{\mu} e^{-x/\mu} \quad \text{for } x \geq 0,\ \mu > 0$$

where: μ = mean

e = 2.71828

Exponential Probability Distribution

- Cumulative Exponential Distribution Function

$$P(x \leq x_0) = 1 - e^{-x_0/\mu}$$

where:

x_0 = some specific value of x

Example: Al's Carwash

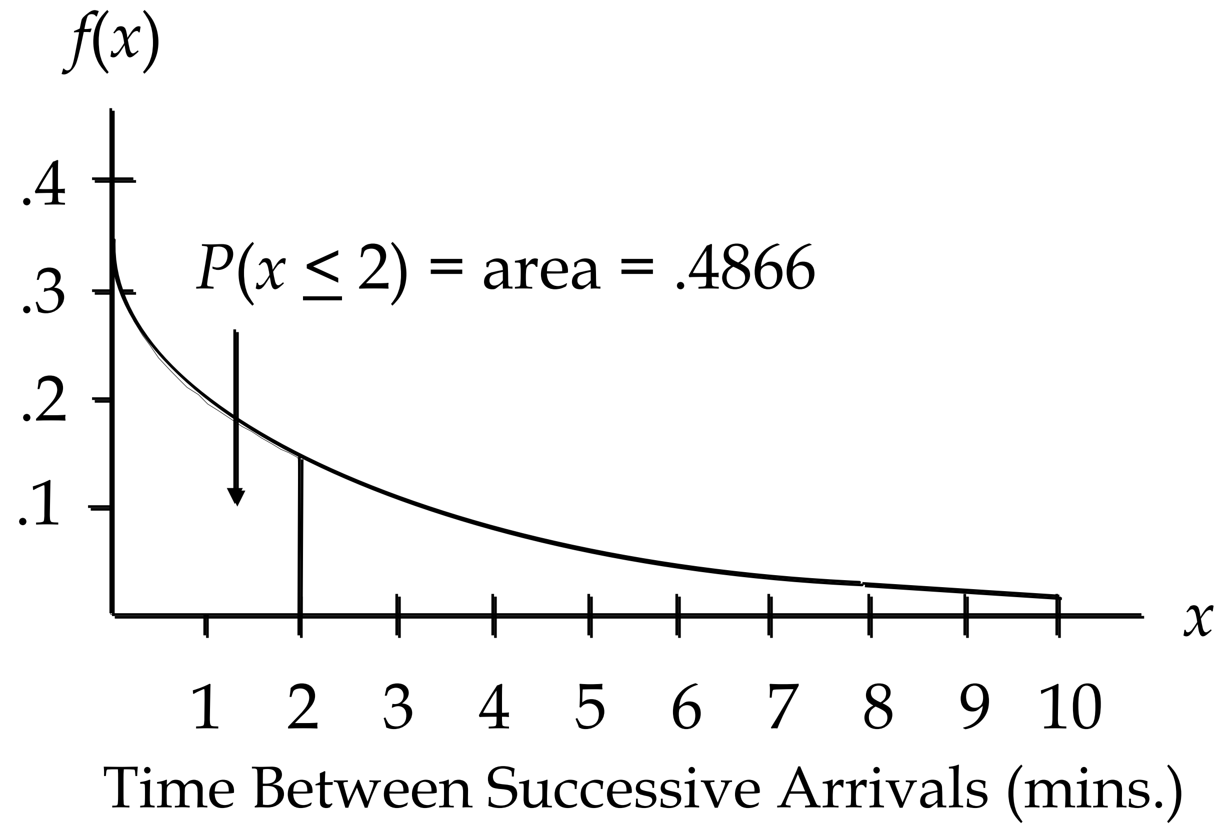
- Exponential Probability Distribution

  The time between arrivals of cars at Al's Carwash follows an exponential probability distribution with a mean time between arrivals of 3 minutes. Al would like to know the probability that the time between two successive arrivals will be 2 minutes or less.

$$P(x \leq 2) = 1 - 2.71828^{-2/3} = 1 - .5134 = .4866$$

Example: Al's Carwash

- Graph of the Probability Density Function



Relationship between the Poisson and Exponential Distributions

(If) the Poisson distribution provides an appropriate description of the number of occurrences per interval

↕

(If) the exponential distribution provides an appropriate description of the length of the interval between occurrences